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CONTENTS

How far away is infinity?

For mathematicians infinity is at ∞,

for physicists ∞ is at the center of the Sun,

and for engineers ∞ is at the center of the Earth.

ON CONDITIONAL FRACTIONAL EQUATIONS IN A FIELD, INVOLVING ONE UNKNOWN

H. S. Vandiver*

Usually in textbooks in high school and college algebras when conditional fractional equations in one unknown are introduced for solution, it is pointed out by the author of such a text that after "clearing of fractions" and solving the resulting integral equations we may find a possible value for the unknown which does not satisfy the original equation. Since in solving any conditional equation we must assume at the beginning that a quantity x exists which satisfies the equation in some set of quantities, say a field, then we have not proved that our equation has any solutions in such a field until we check the possible values found, by substitution in the original equation. That being the case, we need not worry too much about finding that some values do not satisfy, since any such values may be discarded. This will not be too long a process for the fractional equations considered, say, in high schools and colleges; however, we may have problems of this character which lead finally to equations of higher degree than the second. In such cases it is of value to consider a process of obtaining throughout our work only conditional equations which have the same roots in our field as the one with which we started,[1] or, in other words, equations equivalent to the original, as to our field. Otherwise we may have to consider, finally, equations of higher degree than necessary. I have not noticed so far any high school or college text on algebra which solves this problem. Perhaps the absence of such a solution is not surprising since the proof of Theorem I of our present paper requires the use of a theorem involving polynomials, which is not usually found in elementary books, even in the special case when the field involved is the rational one. The question is so elementary that it would be surprising if some one has not disposed of it before this. However, since it is certainly not well-known material, the present article can be considered as expository, if not new.

We shall here prove a rather general theorem, but we begin the discussion by considering a special case. Suppose we inquire as to the possible values of x in the complex field, which satisfy the equation

$$\frac{4x}{x^2-1}-\frac{x+1}{x-1}=1. \qquad (1)$$

We assume there is such an x, and write

$$F(x) = \frac{4x}{x^2-1} - \frac{x+1}{x-1} - 1,$$

or

$$(x^2-1)F(x) = -2(x^2-x).$$

Now $F(x)$ cannot be zero for $x^2-1=0$ or $x=\pm 1$, as is seen when we substitute either of these values in (1), so $(x+1)F(x) = -2x$. Hence the only value of x which makes $F(x) = 0$ is zero, since $x+1 \neq 0$. Hence zero is the only possible solution of (1). That it is a solution is verified by substitution in (1).

In the material that follows we shall use some well-known terms and results[2] concerning polynomials with coefficients in a field, and in addition, the term "least common multiple" (L. C. M.) of a number of polynomials with coefficients in a field. It is the monic polynomial of least degree which is divisible by each of said polynomials, a monic polynomial being one whose leading coefficient is unity.

We shall now consider the general problem. Let $N_i(x)$ for $i = 1, 2, \cdots, k$; $f_i(x)$ for $i = 1, 2, \cdots, k$ be polynomials in an indeterminate x with coefficients belonging to some field $\mathbf{F}$, or in other words, elements of $\mathbf{F}[x]$.

$$F(x) = \frac{N_1(x)}{f_1(x)} + \frac{N_2(x)}{f_2(x)} + \cdots + \frac{N_k(x)}{f_k(x)}, \tag{2}$$

and we then examine the values for which $F(x) = 0$ in $\mathbf{F}$ of x.

Let $M(x)$ be the L. C. M. of $f_i(x)$; $i = 1, 2, \cdots, k$. Let a factorization into irreducible factors of $M(x)$ in $\mathbf{F}[x]$ be

$$M(x) = g_1(x)\ g_2(x)\ \cdots,\ g_r(x)$$

where the g's are monic. This factorization is unique aside from the order of the factors and units as factors. We then note that each of the g's must divide at least one of the f_i's for $i = 1, 2, \cdots, k$. Hence if $g_a(x)$ is zero for some value of x, say x', and for some a in the set $1, 2, \cdots, r$, then $f_b(x')$ is zero for some b in the set $1, 2, \cdots, k$, but if we consider values of x which satisfy (2), when equated to zero, then x' can not satisfy this equation since when we substitute x' in $F(x) = 0$ we obtain at least one zero denominator. If we collect the fractions in (2) we obtain

$$F(x) = \frac{R(x)}{M(x)} \tag{3}$$

where $R(x)$ is in $\mathbf{F}[x]$. Denote the G. C. D of R and M in $\mathbf{F}[x]$ by $D(x)$.

Then write

$$R(x) = D(x)\, T(x).$$

Then for each x in $\mathbf{F}$,

$$M(x)\, F(x) = D(x)\, T(x). \tag{4}$$

If the equation formed by (2) equated to zero is satisfied by $x = x_1$, $x_1 \epsilon \mathbf{F}$, then since $D(x)$ divides $M(x)$, and since every irreducible factor of $M(x)$ divides some f_i, then $D(x_1) \neq 0$. Substituting $x = x_1$ in (4) we have

$$D(x_1)\, T(x_1) = 0,$$

and therefore

$$T(x_1) = 0, \tag{5}$$

which means that any root of $F(x) = 0$ will satisfy $T(x) = 0$. Also, let us assume $T(x_2) = 0$ for some x_2 in $\mathbf{F}$, and we wish to show that $F(x_2) = 0$. From (4) we obtain

$$M(x_2)\, F(x_2) = 0,$$

and since $M(x_2) \neq 0$, then

$$F(x_2) = 0. \tag{6}$$

The relations (5) and (6) show that $F(x) = 0$ and $T(x) = 0$ are equivalent equations as to $\mathbf{F}$, and this proves the following theorem:

Theorem 1. Let $F(x)$ be defined as in (2) *and* (3) *and let $R(x) = D(x)\, T(x)$ where $D(x)$ is the G.C.D. of $R(x)$ and $M(x)$ in $\mathbf{F}[x]$. Then $T(x) = 0$ and $F(x) = 0$ are equivalent equations, as to $\mathbf{F}$.*

We now consider two special cases. If all the coefficients of $T(x)$ are zero except the last say α, an element of $\mathbf{F}$, and if $\alpha \neq 0$, then obviously (2) has no solution. On the other hand, if $\alpha = 0$, then $F(x) = 0$ is satisfied for all values of x except those for which $M(x) = 0$.

Referring again to Theorem 1, we may note that in some particular fractional equations the process of finding the least common denominator of the various denominators given in the equation may be quite long. It will then be more convenient to select in place of our $M(x)$ a polynomial which is a multiple of $M(x)$ but contains only irreducible factors contained in $M(x)$, and in particular the case where $M(x)$ is the product of all the f's. If we do this, we will find that the work will be longer in finding a value of $D(x)$, but this process is direct and does not involve the necessity of factoring all the f's. We may prove a theorem analogous to (1) for this new $M(x)$, the steps in the proof being quite analogous to that of

Theorem 1.

FOOTNOTES

*The author's work on this paper was done under Basic Research Grant 3697, which was awarded to him by the National Science Foundation.

1. E. J. Wilczynski, *College Algebra* (edited by H. E. Slaught), Allyn and Bacon, New York, 1944, pp. 241-244. The author discusses at length the equivalence of fractional rational equations but gives no method for completely avoiding the introduction of values of the unknown quantity, in the process of solution, which do not satisfy the original equation.

2. They will be found in the text by A. A. Albert, *Fundamental Concepts of Higher Algebra*, The University of Chicago Press, Chicago 1956, pp. 48-49.

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NUMERICAL ANALYSIS AND THE DIRICHLET PROBLEM

Donald Greenspan

1. *Introduction.* One of the most exciting mathematical problems, which has such diverse origins as electric field theory, fluid dynamics, heat conduction and elasticity, is the Dirichlet problem. We state it in the following form. If G is a closed, bounded, simply connected plane region whose interior is denoted by R and whose boundary curve is denoted by S, and if $g(x, y)$ is continuous on S, then one must find a function $u(x, y)$, continuous on G, which satisfies

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \equiv u_{xx} + u_{yy} = 0, \text{ on } R; \tag{1.1}$$

and

$$u(x, y) = g(x, y), \text{ on } S. \tag{1.2}$$

The fact that the Dirichlet problem has, under quite general conditions, a unique solution can be demonstrated by a variety of means, including finite differences [10], Dirichlet's principle [31], integral equations [35], subharmonic and superharmonic functions [47], and conformal mapping (Riemann).

The analytical determination of $u(x, y)$, however, is a far more difficult problem than that of establishing its existence. If G is rectangular, then $u(x, y)$ can be given precisely by Fourier series [30], while if G is circular, then $u(x, y)$ can be given in terms of the Poisson integral [47]. Also, any problem for which an explicit conformal map can be constructed which takes G onto a rectangle, or a circle, can be solved. Beyond these cases, the problems involved in producing the analytical solution $u(x, y)$ are overwhelming.

Hence, it is with some amount of delight that one can view the progress towards resolving the general Dirichlet problem which has transpired during the last ten years. The tremendous advances in the field of high speed computation have opened a new avenue of approach and we proceed now to discuss one of the most popular numerical methods now in use.

2. *Numerical Method.* Let h be a fixed, positive constant and let $(\overline{x}, \overline{y})$ be an arbitrary, but fixed, point of G. Denote by G_h the set of all points of the form $(\overline{x}+mh, \overline{y}+nh)$ contained in G, where m and n vary over

all the integers. Two points (x_1, y_1) and (x_2, y_2) of G_h are called adjacent if and only if

(a) the straight line segment joining them is contained entirely in G, and

(b) $(x_2-x_1)^2+(y_2-y_1)^2 = h^2$.

The interior of G_h, denoted by R_h, is the set of all points of G_h which have four adjacent points in G_h. The boundary of G_h, denoted by S_h, and

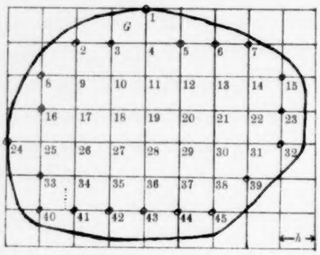

Diagram 1

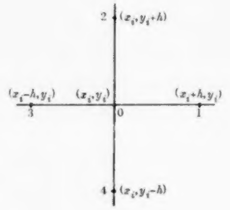

Diagram 2

called the lattice boundary, is defined by $G_h = R_h \cup S_h$, $R_h \cap S_h = \theta$. As an example of these concepts, consider Diagram 1. The points of G_h are there numbered 1-45. The points of R_h are numbered 4, 9-14, 17-22, 25-31, 34-38, while the points of S_h are numbered 1-3, 5-8, 15, 16, 23, 24, 32, 33, 39-45. If the points of S_h are circled, as in the diagram, it is readily apparent why S_h is called the lattice boundary.

In general, if G_h consists of n points, one numbers these in a one-to-one fashion with the integers $1, 2, 3, \cdots, n$. Let the coordinates of the point numbered k be denoted by (x_k, y_k) and the unknown function u at (x_k, y_k) by $u(x_k, y_k) \equiv u_k$.

Now, let (x_i, y_i) be an arbitrary point of S_h. Approximate u_i by $g(x', y')$, where (x', y') is the nearest point of S to (x_i, y_i). If (x', y') is not unique, then choose any one of the set of nearest points and use it. The problem of finding numerical approximations to $u(x, y)$ on S_h is, though crudely done, adequate for present purposes.

It is then required that at each point (x_i, y_i) of R_h, that u satisfy

$$-4u(x_i,y_i)+u(x_i+h,y_i)+u(x_i,y_i+h)+u(x_i-h,y_i)+u(x_i,y_i-h) = 0. \tag{2.1}$$

In terms of the subscript notation, if (x_i,y_i), (x_i+h,y_i), (x_i,y_i+h), (x_i-h,y_i), and (x_i,y_i-h) are labelled 0, 1, 2, 3, 4, respectively, as in Diagram 2, then (2.1) can be rewritten as

$$-4u_0+u_1+u_2+u_3+u_4 = 0. \tag{2.2}$$

Application of (2.1) to each point of R_h yields then a system of linear algebraic equations. If R_h consists of K points, then a system of K linear algebraic equations in K unknowns results. That this is so is readily apparent by considering the application of (2.1) at the points numbered 9 and 18 in Diagram 1, for there then results

$$-4u_9 + u_{10} + u_2 + u_8 + u_{17} = 0, \tag{2.3}$$

$$-4u_{18} + u_{19} + u_{10} + u_{17} + u_{27} = 0. \tag{2.4}$$

In (2.3), the values of u_2 and u_8 are known constants, by the method prescribed for points of S_h, while u_9, u_{10}, and u_{17} are unknowns. In (2.4), u_{10}, u_{17}, u_{18}, u_{19} and u_{27} are all unknowns.

The solution of this linear, algebraic system constitutes the numerical solution on R_h. Before showing that this algebraic system has a unique solution, we make some remarks about the numerical method. Note that the continuous data G, S, and R are replaced by the discrete data G_h, S_h, and R_h, respectively, and the differential equation (1.1) is replaced by the difference equation (2.1). The numerical solution is defined only at the points of G_h and is therefore called a discrete, harmonic function [27].

Finally note that equation (2.1) may be rigorously derived in a variety of ways [25], [42]. However, it is a satisfactory difference analogue of (1.1) even from a heuristic point of view since solving (2.1) for $u(x_i, y_i)$ shows that it is the mean value of $u(x_i+h, y)$, $u(x_i, y_i+h)$, $u(x_i-h, y_i)$, $u(x_i, y_i-h)$, and it is well known that a necessary and sufficient condition for a function to be harmonic is that it satisfy the mean value principle [31].

3. *Uniqueness of the Solution of the Linear Algebraic System.* In this section, one must begin to differentiate between the analytical and the numerical solutions. Let $u(x, y)$ be the analytical solution of the Dirichlet problem described in Section 1 and let $U(x, y)$ be the numerical solution of the method described in Section 2. This implies that at any point (x_i, y_i) of R_h, one must have, from (2.1), that

$$-4U(x_i,y_i) + U(x_i+h,y_i) + U(x_i,y_i+h) + U(x_i-h,y_i) + U(x_i,y_i-h) = 0. \tag{3.1}$$

Theorem 1. The linear algebraic system described in the numerical method of Section 2 possesses a unique solution.

Proof. It is sufficient to show that the determinant of the linear system is not zero. This will be done by demonstrating that the only solution of the homogeneous system which results by considering $g(x, y) \equiv 0$ on S is

the zero vector. Suppose then there exists a non-trivial solution for the homogeneous system. For some point of R_h, then $U \neq 0$. Suppose $U > 0$. (A similar proof follows for $U < 0$.) Let the largest value M of U occur at (x_0, y_0). If the points (x_0+h, y_0), (x_0, y_0+h), (x_0-h, y_0), (x_0, y_0-h) are numbered 1, 2, 3, 4, respectively, then

$$U_0 = \frac{1}{4}[U_1+U_2+U_3+U_4], \tag{3.2}$$

and

$$M = U_0 \geqq U_i, \quad i = 1, 2, 3, 4. \tag{3.3}$$

If $U_1 = U_2 = U_3 = U_4$, then, from (3.2) and (3.3), $U_0 = U_1 = U_2 = U_3 = U_4 = M$. If all of U_1, U_2, U_3, U_4 are not equal, then one is a maximum. Without loss of generality, assume that U_4 is that maximum. Hence

$$U_j = U_4 - c_j, \quad j = 1, 2, 3 \tag{3.4}$$

where c_1, c_2, c_3 are non-negative and at least one is positive. Substitution of (3.4) into (3.2) yields

$$U_0 = U_4 - \frac{c_1+c_2+c_3}{4}$$

from which it follows that $U_4 > U_0$, which contradicts (3.3). Hence, $U_0 = U_1 = U_2 = U_3 = U_4 = M$.

Repeating this argument now at the point numbered 1, and continuing in an inductive fashion, it takes only a finite number of steps to show that $U_d = M$, where d is a number assigned to some point of S_h. Hence $U_d = M > 0$. However, since $g(x, y) \equiv 0$ on S, it follows that $U = 0$ at every point of S_h. Hence $U_d = 0$, which is a contradiction, and the theorem follows.

Some remarks are now in order germane to the actual calculation of the solution vector of the linear system. The solution can, of course, be given explicitly by Cramer's rule in terms of determinants. If the number of equations is "large", say, over three hundred, then the determinants involved are somewhat unwieldy, but one can attempt to evaluate these by means of high speed computation. Other methods, also readily adaptable to today's computational machinery, which can be utilized in attempting to solve the linear system are relaxation [59], over-relaxation [70], matrix inversion [22], gradients, and conjugate gradients [28]. The basic reason for having more than one method available is that there is an accumulation of truncation and roundoff error, inherent in all extensive manipulation on

computers, which may or may not destroy the accuracy of the final results. It appears, from experimentation, that while one method may yield highly inaccurate results for a specific problem, another method may behave almost ideally. The process of accumulation of error is however not well understood at the present time and much research effort is being expended in this direction.

4. *Convergence*. The question of convergence of the numerical solution to the analytical solution as the mesh midth h converges to zero may be treated in the following fashion.

Define the linear operator L by

$$(4.1)\quad L[w(x,y)] \equiv \frac{1}{h^2}\{-4w(x,y)+w(x+h,y)+w(x,y+h)+w(x-h,y)+w(x,y-h)\}.$$

The following lemmas then result [23].

Lemma 1. If $L[v] \leqq 0$ on R_h and $v \geqq 0$ on S_h, then $v \geqq 0$ on R_h.

Lemma 2. If $L[v_2] \leqq -|L[v_1]|$ on R_h and $|v_1| \leqq v_2$ on S_h, then $|v_1| \leqq v_2$ on R_h.

Lemma 3. If $|L[v]| \leqq A$ on R_h, $|v| \leqq B$ on S_h, and r is the radius of any circle which contains G, then on R_h $|v| \leqq B + (Ar^2/4)$.

With the aid of these lemmas, the following theorem can be proved [23].

Theorem 2. If $u(x, y)$ is of class C^4 in G, $u(x, y)$ is the analytical solution of the Dirichlet problem, $U(x, y)$ is the numerical solution described in Section 2, then on R_h

$$(4.2)\qquad |U(x, y) - u(x, y)| \leqq \frac{1}{24} M_4 r^2 h^2 + 2hM_1,$$

where:

$$M_k = \max\left\{ \max_{(x,y)} \epsilon G \left|\frac{\partial^k u}{\partial x^k}\right|,\ \max_{(x,y)} \epsilon G \left|\frac{\partial^k u}{\partial y^k}\right| \right\},$$

r = radius of any circle containing G,

h = mesh width.

The convergence of the numerical to the analytical solution is finally established by

Theorem 3. Under the conditions of Theorem 2, $U \to u$ as $h \to 0$. (The proof for points of R_h follows directly from inequality (4.2), while the proof for points of S_h follows readily from the numerical method prescribed at these points [23].)

A detailed analysis of (4.2) shows that the term $\frac{1}{24} M_4 r^2 h^2$ is the result of approximating (1.1) by (2.1), while the term $2M_1 h$ is the result of

the depraved method for approximating U on S_h. With this in mind, Collatz [8] showed that the term $2M_1h$ could be replaced by a term of the type $O(h^2)$ if one were to use a linear interpolation scheme at the points of S_h rather than the ultra-simple method described in Section 2.

Finally, it should be noted that (4.2) is a "weak" error bound in the sense that it aids in the establishment of Theorem 3, but has no other practical value. Of great significance would be a rule which would enable one to determine h so that for (4.2) one could make

$$|U(x, y) - u(x, y)| < \epsilon$$

where ϵ is an arbitrary, but fixed, tolerance. The presence of the terms M_1 and M_4, however, precludes the use of (4.2) for this purpose, since M_1 and M_4 are functions of $u(x, y)$, and $u(x, y)$ is unknown. Because this is the case, the analyst knows only from (4.2) that, neglecting the ominous presence of accumulation of error, the smaller one selects h, the better are the numerical results. This state of affairs has led to experimentation at centers of numerical analysis research in coding and resolving systems of the order of 1500 linear algebraic equations in 1500 unknowns. Much promise, however, for establishing computable error bounds arises from the recent work of Golomb and Weinberger [71].

5. *Generalizations.* The numerical method, the theorem on the uniqueness of the numerical solution, and the theorem on the convergence of the numerical to the analytical solution can, *with suitable modifications*, be extended in a variety of directions. Numerical methods have been devised which use triangular or rectangular grids [41]. Grids of these types are shown in Diagrams 3 and 4. Of course if $h_1 = h_2 = h$, in Diagram 4, then the grid is called a square grid. In place of using five points, as in difference equation (2.1), difference equations using more points have been constructed [42]. The numerical ideas also extend to Dirichlet type problems, that is, to boundary value problems of the type described in Section

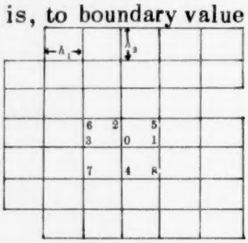

Diagram 3

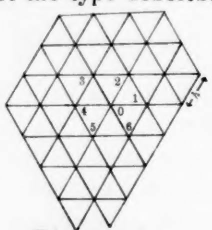

Diagram 4

1 which have the Laplace equation (1.1) replaced by a different elliptic differential equation [23], [24]. Finally it should be noted that the technique extends to n-dimensional problems [23], [65].

To conclude, we give in tabular form, for various generalizations discussed above, the elliptic differential equation and the corresponding difference equation which one could apply at the points (x_i, y_i) of R_h in order to generate the linear algebraic system of the numerical method. The bracketed expressions after each difference equation indicate how many points are used in the difference analogue and the type of grid structure. For simplicity, the point (x_i, y_i) is labelled 0. For triangular grids, refer to Diagram 3, for square grids refer to Diagram 4 with $h_1 = h_2 = h$, and for rectangular grids refer to Diagram 4.

TABLE

Differential Equation	Difference Equation
1. $u_{xx} + u_{yy} = 0$ (Laplace's equation)	1. (a) $-4u_0+u_1+u_2+u_3+u_4 = 0$, [5-point square]
	(b) $-2u_0+\frac{h_2^2}{h_1^2+h_2^2}(u_1+u_3)+\frac{h_1^2}{h_1^2+h_2^2}(u_2+u_4) = 0$, [5-point rectangular]
	(c) $-6u_0+u_1+u_2+u_3+u_4+u_5+u_6 = 0$, [7-point triangular]
	(d) $-20u_0+4(u_1+u_2+u_3+u_4)+u_5+u_6+u_7+u_8 = 0$, [9-point square]
	(e) $-20u_0 + 2\frac{5-p^2}{1+p^2}(u_1+u_3)+2\frac{5p^2-1}{1+p^2}(u_2+u_4)+u_5 + u_6 + u_7 + u_8 = 0$, [9-point rectangular, $p = h_1/h_2$]
2. $u_{xx}+u_{yy}+a(x,y)u_x+b(x,y)u_y + c(x, y)u = G(x, y)$	2. $-(4-c_0h^2)u_0+(1+\frac{a_0h}{2})u_1+(1+\frac{b_0h}{2})u_2+(1-\frac{a_0h}{2})u_3 +(1-\frac{b_0h}{2})u_4 = h^2G_0$, [5-point square]
3. $u_{xx}+u_{yy}-\frac{1}{y}u_y = 0$ (Stokes Stream Equation)	3. (a) $-4u_0 + u_1 + (1-\frac{h}{2y_0})u_2+ u_3+ (1+\frac{h}{2y_0})u_4 = 0$; $y_0 \neq 0$, [5-point square]

(b) $2(\frac{1}{y_0}+\frac{1}{2y_0+h}+\frac{1}{2y_0-h})u_0-\frac{1}{y_0}(u_1+u_3)-\frac{2u_2}{2y_0+h}$
$-\frac{2u_4}{2y_0-h}=0$; [5-point square, $y_0\neq 0$, $0<h<|y|$ for all y in G, equation is symmetric]

(c) $20(-96+4q^2+q^4)u_0-2(-192+44q^2+9q^4)(u_1+u_3)+2(192-96q+28q^2-18q^3+5q^4-2q^5)u_2+2(192+96q+28q^2+18q^3+5q^4+2q^5)u_4-(-96+48q+4q^2-6q^3+q^4)(u_5+u_6)-(-96-48q+4q^2+6q^3+q^4)(u_7+u_8)=0$; [9-point square, $y_0\neq 0$, $q=h/y_0$]

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4. $u_{xx}+u_{yy}-\frac{K}{y}u_y=0$

4. $-2(1+p^{-2})u_0+p^{-2}(u_1+u_3)+(1-\frac{h_2K}{2y_0})u_3+(1+\frac{h_2K}{2y_0})u_4=0$, [5-point rectangular, $y_0\neq 0$, $p={}^{h}{}_1/h_2$]

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5. $\sum_{i=1}^{n}\frac{\partial^2 u}{\partial x_i^{\,2}}=f(x_1,x_2,\cdots,x_n)$ (n-dimensional Poisson equation)

5. $-2u(x_1,x_2,\cdots,x_n)+\left[\sum_{j=1}^{n}B_j\right]^{-1}\sum_{j=1}^{n}\{B_j[u(x_1,x_2,\cdots,x_{j-1},x_j+h_j,x_{j+1},\cdots,x_n)+u(x_1,x_2,\cdots,x_{j-1},x_j-h_j,x_{j+1},\cdots,x_n)]\}=h_1^{\,2}h_2^{\,2}\cdots h_n^{\,2}\left[\sum_{j=1}^{n}B_j\right]^{-1}f(x_1,x_2,\cdots,x_n)$, [$(2n+1)$-point "rectangular"; $B_1=h_2^{\,2}h_3^{\,2}\cdots h_n^{\,2}$, $B_n=h_1^{\,2}h_2^{\,2}h_3^{\,2}\cdots h_{n-1}^{\,2}$; $B_j=h_1^{\,2}h_2^{\,2}\cdots h_{j-1}^{\,2}h_{j+1}^{\,2}\cdots h_n^{\,2}$, $j=2,3,\cdots,n-1$]

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SOME REMARKS ON THE LOGARITHMIC FUNCTION IN THE COMPLEX PLANE

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1. **Introduction.** The customary first course in complex variable does not usually have time to deal with the logarithmic function in any but the briefest fashion. This is unfortunate because it is one of the simplest of the multiple-valued analytic functions and because it is closely related to other multi-valued functions such as $\sqrt{z}$.

In the next section, we prove (without assuming any knowledge of the logarithmic function) that a function element of the logarithm can be analytically continued along each continuous curve not passing through the origin. This is of considerable pedagogical interest because it supplies a concrete example of analytic continuation by power series — a method which is frequently used in defining an analytic function globally.

In the following section, we show, by a suitable choice of the continuous curve along which the analytic continuation is made, how it is possible to return to the starting point with a different function element. This example not only shows how multiple-valued functions can arise from the process of analytic continuation but also furnishes an example of the failure of the monodromy theorem for non-simply connected regions.

The remaining sections deal with $\log z$, $\operatorname{Log} f(z)$ and the relation of these functions to the example of the earlier sections. In doing this, we develop the various properties of these functions in the complex plane and show how these are related to the properties of the functions on the real line. This is done in considerable detail because of the intrinsic importance of these functions, because of the principles which are illustrated by this treatment, and because the standard texts do not seem to be too satisfactory in this regard.

2. **An example of analytic continuation.** Let C be a given continuous curve not passing through the origin; then there is a $\delta > 0$ such that every point on C has a distance from the origin which is at least δ. Let ζ be the initial point and ζ' be the terminal point of C. Let $z_0, z_1, \cdots, z_m$ be points on C such that $z_0 = \zeta$, $z_m = \zeta'$ and $|z_{k+1} - z_k| < \delta$ for $k = 0, 1, \cdots, m-1$; such points exist as a result of the theorem of uniform continuity applied to the continuous curve C.

For a given complex number a_0 we consider the function element

$$(1) \qquad f_0(z) = a_0 - \sum_{n=1}^{\infty} \frac{1}{n}\left(1-\frac{z}{z_0}\right)^n.$$

The series converges if $|1-z/z_0| < 1$ or if $|z-z_0| < |z_0|$; thus, it converges inside the circle about z_0 which passes through the origin. We now show that it is possible to analytically continue $f_0(z)$ along C.

Since z_0 is on C we have $|z_0| \geqq \delta$; hence the series for $f_0(z)$ converges if $|z-z_0| < \delta$. Since $|z_1-z_0| < \delta$, we can write for z in a suitable neighborhood of z_1

$$f_0(z) = f_0(z_1) + \sum_{n=1}^{\infty} b_n(z-z_1)^n, \quad b_n = f_0^{(n)}(z_1)/n!\ .$$

By (1),

$$(2) \qquad f_0'(z) = -\sum_{n=1}^{\infty} \left(1-\frac{z}{z_0}\right)^{n-1}\left(-\frac{1}{z_0}\right) = \frac{1}{z_0}\,\frac{1}{1-(1-z/z_0)} = \frac{1}{z} = z^{-1}.$$

Hence, if $n \geqq 1$

$$b_n = \frac{1}{n!}\left\{\frac{d^{n-1}}{dz^{n-1}} z^{-1}\right\}_{z=z_1} = \frac{1}{n!}\left\{(-1)^{n-1}(n-1)!z^{-n}\right\}_{z=z_1} = -\frac{1}{n}\left(\frac{-1}{z_1}\right)^n.$$

Thus, in some neighborhood of z_1 we have

$$(3) \qquad f_0(z) = f_0(z_1) - \sum_{n=1}^{\infty} \frac{1}{n}\left(-\frac{1}{z_1}\right)^n(z-z_1)^n = f_0(z_1) - \sum_{n=1}^{\infty} \frac{1}{n}\left(1-\frac{z}{z_1}\right)^n.$$

Writing

$$a_1 = f_0(z_1), \quad f_1(z) = a_1 - \sum_{n=1}^{\infty} \frac{1}{n}\left(1-\frac{z}{z_1}\right)^n,$$

we see that $f_1(z)$ is an immediate analytic continuation of $f_0(z)$. Furthermore, this new function is of the same form as the original function element.

Consequently, we can continue this process so as to obtain a function element $f_2(z)$ about z_2, a function element $f_3(z)$ about z_3 and so forth until we finally obtain a function element $f_m(z)$ about z_m; each of these

function elements is an immediate analytic continuation of the preceding one. The general element $f_k(z)$ appears as follows:

$$f_k(z) = a_k - \sum_{n=1}^{\infty} \frac{1}{n}\left(1-\frac{z}{z_k}\right)^n, \qquad a_k = a_0 - \sum_{r=0}^{k-1}\sum_{n=1}^{\infty} \frac{1}{n}\left(1-\frac{z_{r+1}}{z_r}\right)^n.$$

That the above equation holds for a_k is readily established by induction. For, $a_1 = f_0(z_1)$ and thus (1) shows that the above formula holds for $k = 1$. And if it holds for k, then

$$\begin{aligned} a_{k+1} = f_k(z_{k+1}) &= a_k - \sum_{n=1}^{\infty} \frac{1}{n}\left(1-\frac{z_{k+1}}{z_k}\right)^n \\ &= a_0 - \sum_{r=0}^{k-1}\sum_{n=1}^{\infty} \frac{1}{n}\left(1-\frac{z_{r+1}}{z_r}\right)^n - \sum_{n=1}^{\infty} \frac{1}{n}\left(1-\frac{z_{k+1}}{z_k}\right)^n \\ &= a_0 - \sum_{r=0}^{k}\sum_{n=1}^{\infty} \frac{1}{n}\left(1-\frac{z_{r+1}}{z_r}\right)^n \end{aligned}$$

so that the equation holds for $k+1$.

As a result, we have, on taking $k = m$, that

$$f_m(z) = a_0 - \sum_{r=0}^{m-1}\sum_{n=1}^{\infty} \frac{1}{n}\left(1-\frac{z_{r+1}}{z_r}\right)^n - \sum_{n=1}^{\infty} \frac{1}{n}\left(1-\frac{z}{z_m}\right)^n \tag{4}$$

is the result of analytically continuing $f_0(z)$ along the curve C joining $z_0 = \zeta$ and $z_m = \zeta'$. Thus, $f_0(z)$ can be analytically continued along every continuous curve, starting at ζ, which does not pass through the origin.

3. **The example continued.** We now take C to be the unit circle so that $\delta = 1$. We let m be a fixed integer such that $m \geqq 10$ and we define $z_k = e^{2\pi ik/m}$ for $k = 0, 1, \cdots, m$. Then $\zeta = z_0 = 1 = z_m = \zeta'$ and

$$z_{k+1} - z_k = e^{2\pi i(k+1)/m} - e^{2\pi ik/m} = e^{2\pi ik/m}(e^{2\pi i/m} - 1) = e^{2\pi ik/m}\alpha$$

where

$$\alpha = e^{2\pi i/m} - 1 = e^{\pi i/m}(e^{\pi i/m} - e^{-\pi i/m}) = 2ie^{\pi i/m}\sin \pi/m. \tag{5}$$

Hence,

$$(6) \qquad |\alpha| = 2\sin \pi/m = 2\int_0^{\pi/m} \cos t\, dt \leqq 2(\pi/m) = 2\pi/m.$$

Since $m \geqq 7$, we have

$$|z_{k+1} - z_k| = |\alpha| \leqq 2\pi/m < 1 = \delta.$$

Because $z_m = 1 = z_0$ and (4) holds, we have for $|z-1| < 1$

$$f_m(z) = a_0 - \sum_{r=0}^{m-1} \sum_{n=1}^{\infty} \frac{1}{n}\Big(1 - \frac{e^{2\pi i(r+1)/m}}{e^{2\pi i r/m}}\Big)^n - \sum_{n=1}^{\infty} \frac{1}{n}(1-\frac{z}{z_m})^n$$

$$= a_0 - \sum_{n=1}^{\infty} \frac{1}{n}(1-\frac{z}{z_0})^n - \sum_{r=0}^{m-1} \sum_{n=1}^{\infty} \frac{1}{n}(1-e^{2\pi i/m})^n$$

$$(7) \qquad = f_0(z) - m \sum_{n=1}^{\infty} \frac{1}{n}(-\alpha)^n = f_0(z) + m\beta$$

say. Thus, the elements $f_m(z)$ and $f_0(z)$, both about $z_0 = 1$, agree except for the additive constant $m\beta$.

We now show that $\beta \neq 0$ for $m \geqq 10$. Using (6), we obtain

$$|\beta| = \left|\alpha - \sum_{n=2}^{\infty} \frac{1}{n}(-\alpha)^n\right| \geqq |\alpha| - \left|\sum_{n=2}^{\infty} \frac{1}{n}(-\alpha)^n\right| \geqq |\alpha| - \sum_{n=2}^{\infty} \frac{1}{n}|-\alpha|^n$$

$$\geqq |\alpha| - \frac{1}{2}\sum_{n=2}^{\infty} |\alpha|^n = |\alpha| - \frac{|\alpha|^2}{2}\frac{1}{1-|\alpha|} = \frac{|\alpha|}{2(1-|\alpha|)}(2-3|\alpha|)$$

$$\geqq \frac{|\alpha|}{2(1-|\alpha|)}(2-3\cdot\frac{2\pi}{m}) = \frac{|\alpha|}{m(1-|\alpha|)}(m-3\pi) > 0$$

inasmuch as $|\alpha| > 0$ and $m \geqq 10 > 3\pi$. Consequently, $\beta \neq 0$.

Thus, beginning at the point 1, we have analytically continued $f_0(z)$ along the unit circle in the positive direction and have returned to this point with the different function element $f_m(z)$. This has an interesting bearing on the monodromy theorem, one statement of which is as follows:

Let G be a simply-connected region and suppose that the function element $g_0(z)$ about ζ can be analytically continued along every path in G. Then for each given ζ' in G all function elements about ζ',

which arise from the continuation of $g_0(z)$ along different paths in G, agree.

By taking $\zeta' = \zeta$, we see that this result implies that every continuation of $g_0(z)$ from ζ back to ζ along a path in G must yield a function element agreeing with $g_0(z)$.

The above example, at first sight, seems to violate this theorem. Actually, it does not. For if the punctured plane P_0 is defined as the set of all points in the plane with the exception of the origin, then we showed in Section 2 that $f_0(z)$ could be analytically continued along every path in P_0 with initial point ζ. However, P_0 is not a *simply-connected* region even though it is a region. Thus, there is no conflict with the monodromy theorem, and this example shows that the hypothesis that G be simply-connected can not be dropped.

In Section 6 below, we show that if we take $a_0 = 0$ and $z_0 = 1$, then the function $f_0(z)$ of (1) is actually $\log z$. We will also show that for β defined by (7) we have $\beta = 2\pi i/m$.

4. **The logarithmic function.** We assume the usual properties of e^z. In particular, $e^z = 1$ if and only if $z = 2\pi ik$ for some integer k. Since $e^z = e^{z'}$ if and only if $e^{z-z'} = 1$, we see that the first equation is equivalent to the stipulation that $z = z' + 2\pi ik$ for some integer k.

As is well known, every $z \neq 0$ can be written in the polar form $z = |z|e^{i\theta}$ where θ is real. Such a value of θ we call *an* argument of z. Then θ' is another argument if and only if $|z|e^{i\theta} = |z|e^{i\theta'}$ or $i\theta = i\theta' + 2\pi ik$; hence, if and only if $\theta = \theta' + 2\pi k$ where k is an integer. Clearly, there is a unique θ' satisfying this condition and the requirement that $-\pi < \theta' \leq \pi$. This unique θ' we denote by $\arg z$ and we call it *the principal value of the argument of* z. Hence, if $z \neq 0$

$$z = |z|e^{i \arg z}, \qquad -\pi < \arg z \leqq \pi \tag{8}$$

and θ is an argument of z if and only if $\theta = \arg z + 2\pi k$ for some integer k.

For a given number z, any number w such that $e^w = z$ is called *a* logarithm of z. Since $e^w e^{-w} = 1$, we see that $e^w \neq 0$ for each w; hence 0 has no logarithm. But, if $z \neq 0$ and $\log|z|$ denotes the natural logarithm of the positive real number $|z|$, then $w' = \log|z| + i\arg z$ is a logarithm of z. For

$$e^{w'} = e^{\log|z| + i \arg z} = e^{\log|z|}e^{i \arg z} = |z|e^{i \arg z} = z.$$

Furthermore, w is another logarithm of z if and only if $e^w = e^{w'}$ or if and only if for some integer k

$$w = w' + 2\pi ik = \log|z| + i(\arg z + 2\pi k);$$

thus, this holds ifand only if w differs from $\log|z|$ by the product of i and some argument of z. We call *w' the principal value of the logarithm of z* and denote it by $\log z$. Thus, if $z \neq 0$

$$\log z = \log|z| + i \arg z \tag{9}$$

and w is a logarithm of z if and only if $w = \log z + 2\pi ik$ for some integer k. Since $\arg z = 0$ if z is a positive real number, the new notation $\log z$ agrees with the old definition of $\log z$ for positive z.

Not all of the usual properties of real logarithms are possessed by $\log z$ unless we restrict z in some fashion. For example, we prove the following:

(10) $\log 1/z = -\log z$, $\quad \arg 1/z = -\arg z \quad$ if $z \in P_c$

(11) $\log zz' = \log z + \log z'$, $\quad \arg zz' = \arg z + \arg z' \quad$ if $z, z' \in R$

(12) $\log z/z' = \log z - \log z'$, $\quad \arg z/z' = \arg z - \arg z' \quad$ if $z, z' \in R$.

In the above equations, P_c is the cut plane consisting of all points in the plane with the exception of the origin and all negative real numbers; and R is the right half plane consisting of all points $x + iy$ such that $x > 0$.

Before proving (10)-(12) we give some counter-examples to show that some restrictions on z and z' are needed.* For example, (10) fails to hold if $z = -1$ since

$$\log 1/(-1) = \log(-1) = \pi i \neq -\pi i = -\log(-1).$$

And (11) fails to hold even if $z' = z$; for this would imply that $\log z^2 = 2\log z$ whereas (for $z = e^{3\pi i/4}$)

$$\log(e^{3\pi i/4})^2 = \log e^{3\pi i/2} = \log e^{-\pi i/2} = -\pi i/2 \neq 3\pi i/2 = 2(3\pi i/4)$$
$$= 2\log e^{3\pi i/4}.$$

To prove (10) assume that $z \in P_c$ so that $-\pi < \arg z < \pi$. Then

$$z = |z|e^{i \arg z}, \quad 1/z = (1/|z|)e^{-i \arg z};$$

and since $-\pi < -\arg z < \pi$ it follows that $\arg 1/z = -\arg z$. Consequently,

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*Actually (11) holds under less stringent conditions. Let $z \in P_c$ and let $\bar{z}$ be the number conjugate to z. The ray through the origin defined by $\arg w = \arg(-\bar{z})$ and the negative real axis divide the w-plane into two (or possibly one) angular regions. Call the region containing the positive real axis G. Then (11) holds if $z' \in G$. In particular, if $z \in R$, then $R \subset G$.

$$\log 1/z = \log |1/z| + i \arg 1/z = -\log |z| - i \arg z = -\log z.$$

To prove (11) assume that $z, z' \epsilon R$ so that $-\pi/2 < \arg z < \pi/2$ with a similar inequality being satisfied by $\arg z'$. Then

$$zz' = e^{\log z} e^{\log z'} = e^{\log z + \log z'}.$$

Consequently, $\log z + \log z'$ is a logarithm of zz' so that for some integer k

$$\log zz' = \log z + \log z' + 2\pi i k.$$

Taking imaginary parts we find

$$\arg zz' = \arg z + \arg z' + 2\pi k.$$

Hence, we have

$$\begin{aligned} 2\pi|k| &= |\arg zz' - \arg z - \arg z'| \\ &\leqq |\arg zz'| + |\arg z| + |\arg z'| < \pi + \pi/2 + \pi/2 = 2\pi \end{aligned}$$

so that $|k| < 1$; since k is an integer, we obtain $k = 0$. This proves (11).

Finally, (12) is easily proved by using (10) and (11) since if $z' \epsilon R$ so is $1/z'$.

The functions $\arg z$ and $\log z$ have been defined for z in P_0. It is fairly clear that these functions are discontinuous at each point ζ on the negative real axis. For, as z tends to ζ through values lying in the upper half plane, $\arg z$ tends to π; but, as z tends to ζ through values lying in the lower half plane, $\arg z$ tends to $-\pi$. Thus, both functions are discontinuous on the negative real axis and at 0. And it is easily seen that these are the only points of discontinuity; consequently, these functions are continuous in the cut plane P_c.

One question remains: what can be said about the analytic nature of $\log z$ in P_c? In the next section, we prove a general result which implies that $\log z$ is analytic in P_c. In the meantime, we sketch a proof which uses less machinery than the general theorem. Using $\arccos \xi$ and $\arcsin \xi$ to denote the principal values of the inverse functions of ξ, $-1 \leqq \xi \leqq 1$, we have if $z = x + iy$

$$\arg z = \begin{cases} \arccos x/|z| & \text{for } z \text{ in the half plane } y > 0 \\ -\arccos x/|z| & \text{for } z \text{ in the half plane } y < 0 \\ \arcsin y/|z| & \text{for } z \text{ in the half plane } x > 0. \end{cases}$$

Also

$$\frac{\partial}{\partial x} \arccos \frac{x}{|z|} = -\frac{|y|}{|z|^2}, \quad \frac{\partial}{\partial y} \arccos \frac{x}{|z|} = \frac{y}{|y|} \frac{x}{|z|^2} \quad \text{if } y \neq 0$$

$$\frac{\partial}{\partial x}\arcsin\frac{y}{|z|} = -\frac{x}{|x|}\frac{y}{|z|^2}, \quad \frac{\partial}{\partial y}\arcsin\frac{y}{|z|} = \frac{|x|}{|z|^2} \quad \text{if } x \neq 0.$$

From these results one sees by inspection that

$$\frac{\partial}{\partial x}\arg z = -\frac{y}{|z|^2}, \quad \frac{\partial}{\partial y}\arg z = \frac{x}{|z|^2} \quad \text{if } z \in P_c.$$

Also one easily obtains

$$\frac{\partial}{\partial x}\log|z| = \frac{x}{|z|^2}, \quad \frac{\partial}{\partial y}\log|z| = \frac{y}{|z|^2} \quad \text{if } z \neq 0.$$

Thus, the Cauchy-Riemann equations are satisfied in P_c by the function $\log|z| + i\arg z = \log z$. Hence $\log z$ is analytic in P_c and

$$\frac{d}{dz}\log z = \frac{\partial}{\partial x}\log|z| + i\frac{\partial}{\partial x}\arg z = \frac{x}{|z|^2} + i\frac{-y}{|z|^2}$$

$$= \frac{x-iy}{(x+iy)(x-iy)} = \frac{1}{x+iy}$$

$$= 1/z. \tag{13}$$

5. **The general logarithmic function.** We now consider the more general situation in which we seek to define the logarithm of an analytic function $f(z)$ rather than merely the logarithm of z. We begin with the following auxiliary result.

Theorem 1. Let H be an arcwise connected set (i.e. a set such that every pair of its points can be joined by a continuous curve in the set). If $g(z)$ and $h(z)$ are logarithms of $f(z)$ which are continuous in H, then there is a constant k, an integer, such that $g(z) - h(z) = 2\pi ik$ for all z in H.

Proof. Our hypotheses imply that $e^{g(z)} = f(z) = e^{h(z)}$ so that $\{g(z) - h(z)\}/(2\pi i)$ is an integer, say $K(z)$. Since $g(z)$ and $h(z)$ are continuous in H, so is $K(z)$. Let z_0 be some fixed point in H. We now show that $K(z) = K(z_0)$ for each z in H; this will prove the result. For given z, let C be a continuous curve in H joining z_0 to z. Since $K(z)$ is uniformly continuous on C, there are points $z_1, \cdots, z_{n-1}, z_n = z$ such that $K(z)$ varies by less than unity on each part of C determined by the points z_m, z_{m+1}. Thus $|K(z_{m+1}) - K(z_m)| < 1$. Since $K(z_{m+1})$ and $K(z_m)$ are integers we have $K(z_{m+1}) = K(z_m)$. Hence,

$$K(z_0) = K(z_1) = K(z_2) = \cdots = K(z_{n-1}) = K(z_n) = K(z)$$

and the proof is complete.

Theorem 2. Let $f(z)$ be analytic and nowhere 0 in the simply-connected region G containing the point z_0. Then there is a function $L_f(z)$ analytic in G such that

$$e^{L_f(z)} = f(z), \qquad L_f'(z) = f'(z)/f(z), \qquad L_f(z_0) = \log f(z_0).$$

This function is the only logarithm of $f(z)$ which is continuous in G and satisfies the last equation.

Proof. Since $f'(z)/f(z)$ is analytic in G and G is simply-connected, Cauchy's theorem shows that

$$L_f(z) = \int_{z_0}^{z} \frac{f'(\zeta)}{f(\zeta)}\, d\zeta + \log f(z_0) \tag{14}$$

is independent of the path of integration provided the path is in G. Furthermore, it is known that $L_f(z)$ is analytic in G and $L_f'(z) = f'(z)/f(z)$; also $L_f(z_0) = \log f(z_0)$. For all ζ in G we have

$$\begin{aligned}\frac{d}{d\zeta}\{e^{-L_f(\zeta)} f(\zeta)\} &= e^{-L_f(\zeta)} f'(\zeta) + f(\zeta) e^{-L_f(\zeta)}\{-L_f'(\zeta)\} \\ &= e^{-L_f(\zeta)}\{f'(\zeta) - f'(\zeta)\} = 0.\end{aligned}$$

Hence, the fundamental theorem of the integral calculus shows that for each path in the simply-connected region G

$$\begin{aligned}0 = \int_{z_0}^{z} 0 \cdot d\zeta &= \{e^{-L_f(\zeta)} f(\zeta)\}_{z_0}^{z} = e^{-L_f(z)} f(z) - e^{-L_f(z_0)} f(z_0) \\ &= e^{-L_f(z)} f(z) - e^{-\log f(z_0)} f(z_0) = e^{-L_f(z)} f(z) - 1.\end{aligned}$$

Hence, $e^{L_f(z)} = f(z)$. Thus, $L_f(z)$ has all of the properties stated since the uniqueness statement is a consequence of Theorem 1.

The notation $L_f(z)$ is incomplete in the sense that it fails to show its dependence on z_0 and G; nevertheless we use it. However, we do not use the tempting notation $\operatorname{Log} f(z)$ for $L_f(z)$ since it may happen, as shown below, that $f(z) = f(z')$ but $L_f(z) \neq L_f(z')$.

As a first application, we take $f(z) = z$ and $G = P_c$; we show that $L_f(z) = \log z$ so that $\log z$ is analytic in P_c and has derivative $1/z$ thus giving another proof of (13). The simplest proof results from the fact that $\log z$ is a continuous logarithm of z in P_c and takes the value $\log z_0$ at z_0; by the uniqueness part of the theorem, $L_f(z) = \log z$. A second proof, not

using the continuity of $\log z$ in P_c, results from evaluating the integral in (14) by choosing as the path of integration first the line segment $re^{i \arg z_0}$ joining z_0 to $|z|e^{i \arg z_0}$, and second the circular arc $|z|e^{i\phi}$ joining $|z|e^{i \arg z_0}$ to z. Doing this we obtain

$$L_f(z) - \log z_0 = \int_{z_0}^{z} d\zeta/\zeta$$

$$= \int_{|z_0|}^{|z|} \frac{e^{i \arg z_0}\, dr}{re^{i \arg z_0}} + \int_{\arg z_0}^{\arg z} \frac{|z|e^{i\phi} i\, d\phi}{|z|e^{i\phi}}$$

$$= \log|z| - \log|z_0| + i(\arg z - \arg z_0)$$

$$(15) \qquad = \log z - \log z_0.$$

Hence $L_f(z) = \log z$.

In the general case, however, $L_f(z)$ need not coincide with $\log f(z)$ throughout G even though the latter function is a logarithm which takes the value $\log f(z_0)$ at z_0. By Theorem 2, this can only happen if $\log f(z)$ is not continuous in G. To illustrate this possibility, let $G = P_c$ but now take $f(z) = z^2$. Since $\log z$ is discontinuous when $\arg z = \pi$, the function $\log z^2$ is discontinuous when $\arg z = \pi/2$; hence $\log z^2$ is not continuous in G. However, $L_f(z)$ is readily calculated from (14) and (15) as follows:

$$L_f(z) = \int_{z_0}^{z} \frac{2\zeta}{\zeta^2} d\zeta + \log z_0^2 = 2\int_{z_0}^{z} \frac{d\zeta}{\zeta} + \log z_0^2$$

$$= 2(\log z - \log z_0) + \log z_0^2 = 2 \log z + \log z_0^2 - 2 \log z_0.$$

Moreover, this formula shows that $L_f(z)$ does depend on z_0 since the work of the preceding section shows that

$$\log z_0^2 - 2 \log z_0 = \begin{cases} 0 & \text{if } z_0 = 1 \\ -\pi i/2 - 2(3\pi i/4) = -2\pi i & \text{if } z_0 = e^{3\pi i/4}. \end{cases}$$

This result also shows that even though $f(i) = f(-i)$, it is false that $L_f(i) = L_f(-i)$.

We also observe that the requirement that G be simply-connected is essential for the truth of Theorem 2. For, if this result held for all regions we could apply it to the punctured plane P_0 and the function $f(z) = z$.

We would then conclude that $L_f(z)$ is analytic in P_0; hence, $L_f(e^{i\phi})$ is continuous on the interval $[-\pi, \pi]$ and is a logarithm of $e^{i\phi}$. Since $i\phi$ is another continuous logarithm of $e^{i\phi}$, Theorem 1 shows that $i\phi - L_f(e^{i\phi}) = 2\pi ik$ for some fixed integer k and all ϕ in $[-\pi, \pi]$. Since, however, the left side has different values at $-\pi$ and π (because $L_f(e^{i\phi})$ has the same value at these points) there is a contradiction.

We now prove some generalizations of (10)-(12). In what follows, $1/f$, fg, f/g are the functions such that $(1/f)(z) = 1/f(z)$, $(fg)(z) = f(z)\,g(z)$ and $(f/g)(z) = f(z)/g(z)$ respectively. Later we use $f+g$ and $f-g$ for the functions such that $(f+g)(z) = f(z)+g(z)$ and $(f-g)(z) = f(z)-g(z)$.

Theorem 3. Let $f(z)$ be analytic and nowhere 0 in the simply-connected region G containing the point z_0. If $f(z_0) \epsilon P_c$, then for all z in G

$$L_{1/f}(z) = -L_f(z)$$

where both logarithms are determined by the same point z_0.

Proof. Clearly, $1/f(z)$ is analytic and nowhere 0 in G so that $L_{1/f}(z)$ exists. Also

$$e^{-L_f(z)} = 1/e^{L_f(z)} = 1/f(z)$$

and hence $-L_f(z)$ is a continuous logarithm of $1/f(z)$. Finally, by (10)

$$-L_f(z_0) = -\log f(z_0) = \log 1/f(z_0).$$

The theorem now follows from the uniqueness part of Theorem 2.

Theorem 4. Let $f(z)$ and $g(z)$ be analytic and nowhere 0 in the simply-connected region G containing the point z_0. If $f(z_0)$, $g(z_0) \epsilon R$, then

$$L_{fg}(z) = L_f(z) + L_g(z), \qquad L_{f/g}(z) = L_f(z) - L_g(z).$$

Proof. Since $f(z)\,g(z)$ is analytic and nowhere 0 in G, it has a logarithm in G. Also

$$e^{L_f(z)+L_g(z)} = e^{L_f(z)}e^{L_g(z)} = f(z)\,g(z)$$

so that $L_f(z)+L_g(z)$ is a continuous logarithm of $f(z)\,g(z)$ in G. Also (11) shows that

$$L_f(z_0)+L_g(z_0) = \log f(z_0) + \log g(z_0) = \log f(z_0)\,g(z_0)$$

and therefore the first conclusion follows from the uniqueness part of Theorem 2. And the second conclusion follows since

$$L_{f/g}(z) = L_{f(1/g)}(z) = L_f(z) + L_{1/g}(z) = L_f(z) - L_g(z)$$

as a result of Theorem 3.

The following result is of interest even though the hypotheses can be weakened.*

Theorem 5. Let $f(z)$ be analytic and nowhere 0 in the simply-connected region G containing z_0. Let $g(z)$ and $h(z)$ be analytic in G. Let $g(z_0)$ be real and $f(z_0)$ be positive. If $\lambda(z) = e^{g(z)L_f(z)}$ then $L_\lambda(z) = g(z)L_f(z)$ where all logarithms are determined by the same point z_0.

Proof. Clearly, $\lambda(z)$ is analytic and nowhere 0 in G and $g(z)L_f(z)$ is a continuous logarithm of $\lambda(z)$ in G. By Theorem 1 there is an integer k such that for all z in G

$$g(z)L_f(z) - L_\lambda(z) = 2\pi i k.$$

Hence, on taking imaginary parts and setting $z = z_0$, we obtain

$$\begin{aligned} 2\pi k &= \mathbf{I}\{g(z_0)L_f(z_0)\} - \mathbf{I}\{L_\lambda(z_0)\} = \mathbf{I}\{g(z_0)\log f(z_0)\} - \mathbf{I}\{\log\lambda(z_0)\} \\ &= \mathbf{I}\{g(z_0)\}\log|f(z_0)| + \mathbf{R}\{g(z_0)\}\arg f(z_0) - \arg\lambda(z_0) \\ &= 0\cdot\log|f(z_0)| + \mathbf{R}\{g(z_0)\}\cdot 0 - \arg\lambda(z_0) = -\arg\lambda(z_0). \end{aligned}$$

Since the argument on the right side is a principal value, we have $2\pi|k| \leqq \pi < 2\pi$ so that $|k| < 1$ or $k = 0$. This proves the result.

6. The series for the logarithmic function. Let $z_0 \epsilon P_c$. Since $\log z$ is analytic in P_c we have in a suitable neighborhood of z_0 that

$$\log z = \log z_0 + \sum_{n=1}^{\infty} c_n(z-z_0)^n, \quad c_n = \frac{1}{n!}\left\{\frac{d^n}{dz^n}\log z\right\}_{z=z_0}.$$

Since $\log z$ has the derivative $1/z$, (2) and (3) show that in some neighborhood of z_0

$$\log z = \log z_0 - \sum_{n=1}^{\infty} \frac{1}{n}\left(1-\frac{z}{z_0}\right)^n.$$

This result holds for all z inside the largest circle about z_0 whose interior is contained in P_c. Hence

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*The proof shows that the hypotheses on $f(z_0)$ and $g(z_0)$ can be replaced by the condition

$$|\mathbf{I}\{g(z_0)\}\log|f(z_0)| + \mathbf{R}\{g(z_0)\}\arg f(z_0)| < \pi .$$

$$(16)\qquad \log z = \log z_0 - \sum_{n=1}^{\infty} \frac{1}{n}\left(1-\frac{z}{z_0}\right)^n \quad \text{if} \begin{cases} |z-z_0| < |z_0| & \text{when } z_0 \in R \\ |z-z_0| < |\mathrm{I}(z_0)| & \text{when } z_0 \notin R. \end{cases}$$

In particular, taking $z_0 = 1$

$$(17)\qquad \log z = -\sum_{n=1}^{\infty} \frac{1}{n}(1-z)^n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(z-1)^n \quad \text{if } |z-1| < 1.$$

Examination of (1) shows that $f_0(z) = \log z$ in a neighborhood of z_0 provided we take $a_0 = \log z_0$. From (7), (5) and (17) we also see that

$$\beta = -\sum_{n=1}^{\infty} \frac{1}{n}(1-e^{2\pi i/m})^n = \log e^{2\pi i/m} = 2\pi i/m$$

provided $|e^{2\pi i/m}-1| < 1$ and $m \geqq 2$. Since (6) shows that the first inequality holds when $m \geqq 7$, we have proved that $\beta = 2\pi i/m$ for $m \geqq 7$. (Actually, this also holds for $m = 6$.)

We are now in a position to understand the failure of the analytic continuation $f_m(z)$ of $f_0(z)$ to coincide with $f_0(z)$. Taking $z_0 = 1$ and $a_0 = 0$, $\log z$ and $f_0(z)$ coincide in a circle about z_0. As long as the path along which we continue $f_0(z)$ remains in P_c, this continuation has to agree with $\log z$ by the theorem on analytic continuation. However, as soon as the negative real axis is crossed, this is no longer the case because $\log z$ is not continuous across this axis whereas the continuation is.

7. Related functions. If $\alpha \neq 0$, we can define α^β to be $e^{\beta \log \alpha}$. It is easy to verify that $\alpha^\beta \alpha^{\beta'} = \alpha^{\beta+\beta'}$ and α^{-1} is the reciprocal of α. From this one easily obtains for positive integral n that the product with n factors $\alpha \cdot \alpha \cdots \alpha$ is just α^n; hence, also, $\alpha^{-n} = 1/(\alpha \cdot \alpha \cdots \alpha)$. Therefore, the new notation agrees with the old definition of α^m for integral m; the new notation also agrees with the old definition when β is real and α is positive.

In particular, $\alpha^{f(z)} = e^{f(z) \log \alpha}$ is analytic in a region G if $f(z)$ is analytic there; thus, α^z is entire. And $z^\beta = e^{\beta \log z}$ is analytic in P_c since $\log z$ is analytic there. The binomial theorem, which gives the Taylor series for $(z+z_0)^\beta$, is now readily obtained from the expansion theorem for analytic functions in the form

$$(18) \qquad (z+z_0)^\beta = \sum_{n=0}^{\infty} \binom{\beta}{n} z_0^{\beta-n} z^n \quad \text{if} \begin{cases} |z| < |z_0| & \text{when } z_0 \in R \\ |z| < |I(z_0)| & \text{when } z_0 \notin R. \end{cases}$$

Also, the functions α^z and z^β are special cases of the function $P_f^g(z)$ defined to be $e^{g(z)L_f(z)}$; this reduces to $f(z)^{g(z)}$ if $L_f(z) = \log f(z)$. The new function is analytic in G if both $f(z)$ and $g(z)$ are, and if $f(z)$ is nowhere 0 in G. With $\lambda(z)$ defined as in Theorem 5, this and Theorem 4 yield the following results under suitable restrictions:

$$P_f^g(z)\,P_f^h(z) = P_f^{g+h}(z), \qquad P_f^g(z)/P_f^h(z) = P_f^{g-h}(z)$$

$$P_f^h(z)\,P_g^h(z) = P_{fg}^h(z), \qquad P_f^h(z)/P_g^h(z) = P_{f/g}^h(z)$$

$$P_\lambda^h(z) = P_f^{gh}(z).$$

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TEACHING OF MATHEMATICS

Edited by

Joseph Seidlin and C. N. Shuster

This department is devoted to the teaching of mathematics. Thus articles on methodology, exposition, curriculum, tests and measurements, and any other topic related to teaching, are invited. Papers on any subject in which you, *as a teacher*, are interested, or questions which you would like others to discuss, should be sent to *Joseph Seidlin, Alfred University, Alfred, New York.*

S. I. M. N. T.

D. H. Hyers

INTRODUCTION: After studying, writing, teaching, and editing mathematical manuscripts for many years, I'm calling for the mathematicians of the United States to unite in an effort to clarify various portions of the language of mathematics. Here the term "mathematics" is not used in some idealized Platonic sense, but in the sense of our everyday mathematical activities, such as teaching and writing.

The problem as I see it is that of efficiency in mathematical education. We professors keep moaning about how stupid our students are, often without probing further into the problem.

According to Mortimer J. Adler and other scholars, mathematics is one of the easier disciplines, as compared with history or philosophy, for example. Why then do many students in high school and college consider it so difficult? Several books could undoubtedly be written in attempting to explain this paradox. It seems that there are several reasons. First, mathematics, like music, is "easy" in the sense that one can do good work in it very early in life. It requires very little or no experience of the world, at least in its "pure" phases. Music has Mozart and Mathematics has Galois, to cite two extreme examples. But this very fact means that we should (and often do) require *more mastery* of the subject by the student in mathematics than one could expect in social studies.

Another peculiarity of mathematics is its "you either get it or you don't" aspect. The normal curve seems to be rather inappropriate to use in connection with grades in mathematics classes. Over a period of years it is apparent from my own experience that the distribution tends to be much simpler than the normal curve. Actually it is more often "trapezoidal", with about the same number of B's, C's, and D's, and about the same number of A's, as F's.

A third reason might be that mathematicians who teach, as well as those who write textbooks, insist on remaining "in the (wrong) groove". The history of the subject is so long and complicated that much of the terminology and notation has "grown like Topsy," and has not been well thought through and codified. Twenty-five years ago, with such a small mathematical population in the United States, this was not so important. But today, in the post-sputnik era, we must standardize *carefully thought out* notations and terminology, if we are going to succeed in producing enough well trained mathematicians, engineers, and physical scientists, to say nothing of the other myriad appliers of mathematics in the modern world. This third aspect of the "paradox" is our thesis.

1. *HIGH SCHOOL MATH.*

Fortunately, the mathematical educators have re-awakened and are busily at work in this field (see e.g., reference [1]). This is a very large subject which is currently being discussed almost everywhere in the United States. The only point which will be mentioned here concerns one of the many experimental programs now under way. The committee headed by Dr. Max Beberman at the University of Illinois has come up with some interesting ideas concerning the teaching of beginning algebra. (The program has been tried at several high schools, but the complete results of the experiments will not be made public until later, as I understand it). In this program, the old terminology is being radically altered, particularly with respect to "equation", "unknown", etc. A conditional equation is to be called a "sentence" and the word "unknown" is to be replaced by "pro-numeral". These words do (or should) help to convey the correct meaning of the terms to the student, instead of vague or ambiguous ideas. At present, some of the latter are gradually understood by the more persistent students only after several months or even years of struggle. The average student never grasps them. Along this same line, we as mathematics teachers might be able to modify the standard notations slightly and come out with much clearer ones. For example, if we look at high school and freshman college mathematics, we observe that there are at least four common uses of the equality sign. The first is "logical identity", essentially a symbol for the English word "is", as in

$$2+3=5. \tag{1}$$

The second occurs in the following problem. Solve the equation

$$\sqrt{x+14}-\sqrt{x-2}=8. \tag{2}$$

Here we are not making a statement as in equation (1), but are asking two questions: are there any values of x which satisfy the condition (2), and if so, what are they? In this case, of course, the answer to the first

part of the question is "no". Perhaps we could better emphasize the nature of such problems by using the symbol $\overset{?}{=}$ obtained by superimposing a question mark on the equality sign, and omitting the dot on the question mark: $\sqrt{x+14}-\sqrt{x-8} \overset{?}{=} 8$.

A third use is for identities such as $(a+b)^2 = a^2+2ab+b^2$ or $\sin^2 A + \cos^2 A = 1$. Here we should remember to write $\equiv$ instead of $=$, to remind the student that these are to be true for all values of the variables. Many teachers do this now, of course.

A fourth use occurs in analytic geometry as well as in the second year of algebra, where the student is asked, for instance, to "graph the curve $x^2+2y^2 = 4$". In addition to the poor mathematical grammar contained in this instruction, the student is also confronted with what amounts to still another use of the equality sign. What is meant, of course, is to plot a suitable number of points (x, y) where y is related to x in accordance with the given equation, and then to draw a curve through these points to give a graphical picture of the *relationship*. It's not completely clear just how we should amend the notation in this case and any suggestions on the part of the readers of this article will be welcome. Should we use: $\overset{r}{=}$, where r stands for "relation"?

2. *COLLEGE AND ADVANCED MATHEMATICS*

With regard to the teaching of calculus, Karl Menger [2] seems to be one of the few who have made a careful analysis of the whole problem of a suitable modern notation for functions, derivatives, integrals, etc., and then followed through to do something about it. As he indicates, the time honored Leibnitz notation certainly tends to obscure the basic ideas. It seems to me that everyone who plans to write a calculus text now or in the future must pay some attention to the Menger's program, even if he doesn't adopt it in toto.

A practical difficulty in this program might occur in bridging the gap between elementary calculus and elementary physics. On the other hand, as many experienced calculus teachers will admit, the present traditional system leaves much to be desired. After the basic ideas are established with the aid of Menger's approach, it should be easy for the student to go over to the traditional Leibnitz notation. It is always easier to learn a "second language" if one knows one language well, particularly if the second language is "easier" in the sense of requiring less new notation than the first.

In the traditional approach, how many students really understand the several meanings of the single symbol dx? Most of them get completely confused. Others encounter the famous book by S. P. Thompson [3] and learn that dx is "a little bit of x". Of course they carefully avoid telling the math. teacher of this heresy, but with this idea they can, perhaps,

understand the physics instructor's use of the calculus.

Perhaps some readers may be laboring under the pleasant delusion that everything is fine with respect to notation and terminology in the *higher* branches of the Queen of the Sciences. If so, one "horrible" ex- suffice to jar him loose from his complacency.

Consider the plight of the student in an advanced seminar in analysis, who must read the literature over a period of a quarter of a century or more. The word "compact" is bound to occur many times. It may be a maturing experience for the student to analyse all the meanings of this one simple word in a random sampling of papers in analysis and topology during the years 1934-59, but it is also a serious waste of time. In my opinion the topologists and analysts together have erected a "Tower of Babel" concerning this one word, and needlessly so.

CONCLUSION: At present we are in the same state as the automotive engineers were shortly after the beginning of this century. They finally agreed on some standard thread sizes, which simplified their lives considerably. With the growing mathematical population, we must do likewise.

The statisticians have taken the lead here and we should follow suit.

To implement this program, we suggest the formation of the Society for the Improvement of Mathematical Notation and Terminology. So far this is a paper organization. Perhaps we should keep it that way. To join, send a paper on this subject (preferably short) to Glenn James, Managing Editor of this Magazine.

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A NOTE ON THE SOLUTION OF SIMULTANEOUS LINEAR EQUATIONS

Glenn D. James

Engineers have been observed solving linear equations by Cramer's Rule as a practical applied method. It is hoped that this note may prove valuable in presenting a more useful method to classes. An explanation of the steps taken follows the reduction:

$$+.322x - 3.96y + 4.73z = 6.543 \tag{1}$$

$$+.527x + 3.07y - 6.50z = 4.746 \tag{2}$$

$$-.209x + 3.43y + 5.62z = 5.381 \tag{3}$$

$$\begin{array}{r} \\ (-)\ .6110 \\ (+)\ 1.5407 \end{array}\begin{pmatrix} +.322 - 3.96 + 4.83 & 6.543 \\ +.528 + 3.08 - 6.50 & 4.746 \\ -.209 + 3.43 + 5.62 & 5.381 \end{pmatrix} \qquad \begin{array}{r} 4 \\ 5 \\ 6 \end{array}$$

$$\begin{pmatrix} +.322 - 3.960 + 4.730 & 6.5430 \\ -.322 - 1.876 + 3.972 - 2.8998 \\ -.322 + 5.285 + 8.659 + 8.2905 \end{pmatrix} \qquad \begin{array}{r} 7 \\ 8 \\ 9 \end{array}$$

$$\begin{array}{r} \\ (+)\ 4.405 \end{array}\begin{pmatrix} -5.836 + \ \ 8.702 + \ \ 3.6432 \\ +1.325 + 13.389 + 14.8335 \end{pmatrix} \qquad \begin{array}{r} 10 \\ 11 \end{array}$$

$$\begin{pmatrix} -5.836 + \ \ 8.702 + \ \ 3.6432 \\ +5.836 + 58.979 + 65.3416 \end{pmatrix} \qquad \begin{array}{r} 12 \\ 13 \end{array}$$

$$\begin{pmatrix} 67.781 + 68.9849 \end{pmatrix} \tag{14}$$

From 14

$$+67.681z = +68.9849 \tag{15}$$

$$\underline{z = +1.019} \tag{16}$$

From 12

$$-5.836y + 8.702z = 3.6432 \tag{17}$$

$$-5.836y = -5.224 \tag{18}$$

$$\underline{y = +\ .8951} \tag{19}$$

From 7

$$+.322x - 3.960y + 4.730z = +\ 6.5430 \tag{20}$$

$$+.322x = +\ 5.268 \tag{21}$$

$$\underline{x = +16.36} \tag{22}$$

Explanation and procedure: The solution is an extension of elimination by subtraction and addition for two unknowns. The three equations, (1, 2 and 3) are copied over in lines 4, 5 and 6 without the unknowns or the equal sign and the matrix symbol placed around the numbers to indicate the grouping. In line 5 the (–) .6110 is minus so that the sign of the leading number (+ .527) will become opposite that of + .322 when the multiplication of line 5 is carried out (i.e. equation 2 is multiplied by (–) .322/.527 = (–) .6110. It is important that signs be handled as a separate step if larger order equations are to be solved without error. 1.5407 in line 6 is .322/.209. Lines 8 and 9 are obtained from 5 and 6 by multiplying through by the constants to the left. Lines 10 and 11 are obtained by adding the members of lines 7 and 8 and lines 7 and 9, respectively. The 4.405 in line 11 is obtained in a similar manner to .6110 in line 5, i.e. 4.405 = 5.836/1.325. Line 13 is obtained from line 11 by multiplication by (+) 4.405, and line 14 by adding lines 12 and 13. Since desk calculators can add and subtract products, line 18 is obtained directly from line 17 as one machine operation.

Remarks: It is important to note that no steps are combined beyond that of a single thought process or calculator operation, i.e. algebraic additions are performed, not algebraic subtractions so that direct machine addition will take care of signs. Also note that the signs are handled as a separate thought in order to insure their proper handling, this has been found extremely important in avoiding errors in working with higher order equations. The method with desk calculators is quite applicable to about 8 equations, after which computors should be used. Computors are programmed in what amounts to the same method.

Since the above method may not be useful in classroom instruction, inasmuch as desk calculators are not generally available to students, consider the following, where the calculations are explained below.

$$+\ 2x + y + 0 = 7 \qquad 23$$

$$+\ 18x + 0 + 12z = 30 \qquad 24$$

$$-\ x + 4y + 3z = -5 \qquad 25$$

or

$$\begin{array}{cc} & \begin{array}{cccc} y & x & z & \end{array} \\ (-)\,4 & \begin{pmatrix} +4 & -1 & +3 & -\ 5 \\ +1 & +2 & +0 & +\ 7 \\ +0 & +9 & +6 & +15 \end{pmatrix} \end{array} \qquad \begin{array}{l} 26\ (25) \\ 27\ (23) \\ 28\ (24) \end{array}$$

$$\begin{pmatrix} +4 & -1 & +3 & -\ 5 \\ -4 & -8 & -0 & -28 \\ +0 & +9 & +6 & +15 \end{pmatrix} \qquad \begin{array}{l} 29 \\ 30 \\ 31 \end{array}$$

$$\begin{pmatrix} -9+3-33 \\ +9+6+15 \end{pmatrix} \qquad \begin{matrix} 32 \\ 33 \end{matrix}$$

$$\begin{pmatrix} +9-18 \end{pmatrix} \qquad 34$$

$$9z = -18 \qquad 35$$

$$\underline{z = -2} \qquad 36$$

$$-9x + 3z = -33 \qquad 37$$

$$-9x = -27 \qquad 38$$

$$\underline{x = +3} \qquad 39$$

From 29

$$+4y - x + 3z = -5 \qquad 40$$

$$+4y = +4 \qquad 41$$

$$\underline{y = +1} \qquad 42$$

Explanation: The matrix is formed from lines 23, 24 and 25 as indicated in ( ) to the right of lines 26, 27 and 28, the x, y and z terms have also been re-arranged to make use of the 0 coefficient. The -4 to the left of line 23 is to be multiplied by the line so that lines 29 and 30 may be added to give line 32. Line 33 is merely line 31 copied since its coefficient in column 1 is seen to be 0 without reduction. Line 34 is line 32 plus line 33. (It may be desired to think of lines as matrices and speak of matrix addition, and multiplication instead of "line" addition.)

There is a tendency to skip lines 26, 27, 28, 29, 30, 31, 34, 37, and 40. This skipping does not seem to be conducive to accurate work in realistic problems and leads to extreme difficulty in avoiding errors when working with more unknowns, it does, however, help in getting the point across when comparing with Cramer's Rule.

If the leading coefficient in line 26 (the $+4$) is made to be divisible by the least common multiple of the leading coefficients of the remaining lines (the $+1$ of 27 and the 0 of 28) then there will be no divisions involved in the reduction for any numbers, i.e. this would correspond to the regular intermediate algebra method.

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MISCELLANEOUS NOTES

Edited by

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Articles intended for this department should be sent to *Charles K. Robbins, Department of Mathematics, Purdue University, Lafayette, Ind.*

A PROOF OF EULER'S LIMIT FROM A WELL KNOWN PHYSICAL PRINCIPLE

Robert E. Shafer

There are many proofs which show that

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x.$$

The following proof should be within the capabilities of the undergraduate student. We shall make use of Newton's first law of motion.

I. Every body will continue in its state of rest or of uniform motion in a straight line except in so far that it is compelled to change that state by impressed force.

Suppose that a particle travels at a velocity v_0. Then in t units of time, the particle has moved a distance $v_0 t$.

Next, suppose this particle is subjected to a retarding force which is proportional to the square of its velocity. Suppose at time $t = 0$, $v = v_0$, and $s = 0$. Then

$$\frac{d^2s}{dt^2} = \frac{dv}{dt} = -kv^2, \quad \text{or} \quad \frac{dt}{dv} = -\frac{1}{kv^2},$$

Solving the second equation, $t = \frac{1}{kv} + L$. At time $t = 0$, $L = -\frac{1}{kv_0}$; so that by substituting $v = \frac{ds}{dt}$,

$$k\frac{ds}{dt} = \frac{kv_0}{1+kv_0 t}$$

which, integrated, yields

$$s = \log(1+kv_0 t)^{1/k} + M$$

At time $t = 0$, the constant of integration M is zero, since $s = 0$. Now if

the constant of proportionality k approaches zero, we have Newton's first law. That is

$$\lim_{k\to 0} \log(1+kv_0t)^{1/k} = v_0t,$$

or

$$\lim_{k\to 0} (1+kv_0t)^{1/k} = e^{v_0t}$$

which is equivalent to Euler's limit.

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A NOTE ON A SIMPLE MATRIX ISOMORPHISM

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It is well known that the correspondence

$$(1) \qquad a+bi \leftrightarrow \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

provides an isomorphism between the field of complex numbers and a real subsystem of the 2-by-2 matrices over the complex field. If f is a function of a matrix arising from a scalar function (see e.g. [2]), the following question is asked: referring to correspondence (1), under what condition does

$$(2) \qquad f(a+bi) \leftrightarrow f\begin{pmatrix} a & b \\ -b & a \end{pmatrix}?$$

This question is answered very simply as follows. Let $\alpha = a+bi$, and let

$$(3) \qquad A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}, \quad E = \begin{pmatrix} 1/2 & 1/2i \\ -1/2i & 1/2 \end{pmatrix}, \quad \text{and} \quad \overline{E} = \begin{pmatrix} 1/2 & -1/2i \\ 1/2i & 1/2 \end{pmatrix}.$$

Since $(f(\alpha)+\overline{f(\alpha)})/2$ and $(f(\alpha)-\overline{f(\alpha)})/2i$ are, respectively, the real and imginary parts of $f(\alpha)$, then $f(\alpha) \leftrightarrow f(\alpha)E + \overline{f(\alpha)}\overline{E}$. On the other hand, since A has spectral form $A = \alpha E + \overline{\alpha}\overline{E}$, then by the definition of a function of a matrix, $f(A) = f(\alpha)E + f(\overline{\alpha})\overline{E}$. From these results it is clear that the correspondence (2) holds if and only if $f(\overline{\alpha}) = \overline{f(\alpha)}$.

This note is concerned with the following generalization. Let C_n be the collection of n-by-n matrices over the complex numbers. Corresponding to each matrix u of C_n, let $U = \phi(u)$ be the matrix of C_{2n} obtained by replacing each element α of u by the corresponding 2-by-2 matrix A of (3). Let S_{2n} be the collection of all such matrices U. Then by means of (1) and block addition and multiplication of matrices, it is easily verified that the mapping ϕ is a ring isomorphism of C_n onto S_{2n}. Again the question is asked: if f is a scalar function for which the matrices $f(u)$ and $f(U)$ are defined, under what condition does $\phi(f(u)) = f(\phi(u))$? This question is answered by the following theorem.

Theorem. *Let $U = \phi(u)$. Let $\alpha_1, \cdots, \alpha_r$ be the distinct eigenvalues of*

u with respective indices $s_1, \cdots, s_r$. Let f be a scalar function for which the matrices $f(u)$ and $f(U)$ are defined. Then, $\phi(f(u) = f(\phi(u))$ if and only if $f^{(k)}(\overline{a}_i) = \overline{f^{(k)}(a_i)}$ for all i and k such that $0 \leq k < s_i$.*

Before proceeding with the proof of this theorem, some preliminary remarks are noted and a lemma is demonstrated.

Let $F = \dot{+}E$ be the direct sum in C_{2n} of n matrices E, where E is given in (3). Similarly, let $\overline{F} = \dot{+}\overline{E}$. Clearly, $F^2 = F$, $\overline{F}^2 = \overline{F}$, $F\overline{F} = 0 = \overline{F}F$, $F + \overline{F} = I$, and $FU = UF$, $\overline{F}U = U\overline{F}$ for each U in S_{2n}. Furthermore, if $a1$ is a scalar matrix in C_n, then $\phi(a1) = aF + \overline{a}\overline{F}$.

Lemma. *Let $a_1, \cdots, a_r$ be the distinct eigenvalues of u with associated indices $s_1, \cdots, s_r$ and principal idempotents $e_1, \cdots, e_r$. Let $E_i = \phi(e_i)$. Then the eigenvalues of $U = \phi(u)$ are $a_1, \overline{a}_1, \cdots, a_r, \overline{a}_r$.** Also, if $a_i \neq \overline{a}_i$, then s_i is the index of each and FE_i and $\overline{F}E_i$ are their respective principal idempotents. If $a_i = \overline{a}_i$, then this eigenvalue is of index s_i with principal idempotent $E_i = FE_i + \overline{F}E_i$.*

Proof. Since $ue_i = e_iu$, then $UE_i = E_iU$, and $UFE_i = FUE_i = FE_iU$. Similarly, $U\overline{F}E_i = \overline{F}E_iU$. Also, $\phi(e_i(u-a_i1)) = E_i(U-a_iF-\overline{a}_i\overline{F}) = FE_i(U-a_iI) + \overline{F}E_i(U-\overline{a}_iI)$. Since $e_i(u-a_i1)$ is nilpotent of index s_i and $F\overline{F} = 0$, both $FE_i(U-a_iI)$ and $\overline{F}E_i(U-\overline{a}_iI)$, (and also $E_i(U-a_iI)$ if $a_i = \overline{a}_i$) are nilpotent of index s_i. Since $\Sigma_{i=1}^r e_i = 1$, then $\Sigma_{i=1}^r E_i = \Sigma_{i=1}^r FE_i + \Sigma_{i=1}^r \overline{F}E_i = I$. Finally, since $e_i^2 = e_i$, then $E_i^2 = E_i$ and $(FE_i)^2 = FE_i$ and $(\overline{F}E_i)^2 = \overline{F}E_i$. The lemma is a consequence (see [3]) of these results.

Proof of the theorem. By the definition of a function of a matrix, (using the notation of the lemma)

$$f(u) = \sum_{i=1}^{r} e_i \sum_{k=0}^{s_i-1} (1/k!)f^{(k)}(a_i)(u-a_i1)^k.$$

Hence, $\phi(f(u))$ is given by

$$\sum_{i=1}^{r}(FE_i+\overline{F}E_i)\sum_{k=0}^{s_i-1}(1/k!)(f^{(k)}(a_i)F+\overline{f^{(k)}(a_i)}\overline{F})(U-a_iF-\overline{a}_i\overline{F})^k$$

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*An index of an eigenvalue is its multiplicity in the minimum polynomial of the matrix.

**This result on the eigenvalues was previously announced by J. W. Ellis in an abstract [1].

$$= \sum_{i=1}^{r} FE_i \sum_{k=0}^{s_i-1} (1/k!) f^{(k)}(a_i)(U-a_iI)^k + \sum_{i=1}^{r} \overline{F}E_i \sum_{k=0}^{s_i-1} (1/k!) \overline{f^{(k)}(a_i)}(U-\overline{a}_iI)^k.$$

On the other hand, (even if $a_i = \overline{a}_i$, since $E_i = FE_i + \overline{F}E_i$), $f(U)$ has the value

$$\sum_{i=1}^{r} FE_i \sum_{k=0}^{s_i-1} (1/k!) f^{(k)}(a_i)(U-a_iI)^k + \sum_{i=1}^{r} \overline{F}E_i \sum_{k=0}^{s_i-1} (1/k!) f^{(k)}(\overline{a}_i)(U-\overline{a}_iI)^k.$$

Clearly, if $f^{(k)}(\overline{a}_i) = \overline{f^{(k)}(a_i)}$ for all i and k such that $0 \leqq k < s_i$, then $\phi(f(u)) = f(\phi(u))$. Conversely, if $\phi(f(u)) = f(\phi(u))$, then by multiplying the difference of the above two representations of $f(U)$ by $\overline{F}E_i$,

$$\overline{F}E_i \sum_{k=0}^{s_i-1} (1/k!)(f^{(k)}(\overline{a}_i) - \overline{f^{(k)}(a_i)})(U-\overline{a}_iI)^k = 0.$$

Since the system $\overline{F}E_i$, $\overline{F}E_i(U-\overline{a}_iI)$, $\dots$, $\overline{F}E_i(U-\overline{a}_iI)^{s_i-1}$ is linearly independent, then $f^{(k)}(\overline{a}_i) = \overline{f^{(k)}(a_i)}$ for all i and k such that $0 \leqq k < s_i$.

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3. J. H. M. Wedderburn, Lectures on Matrices, Amer. Math. Soc. Colloq. Publ. XVII (1934), 27-30.

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ON FINITE SETS AND THE PEANO POSTULATES

D. A. Kearns

There are two common characterizations of finite and of infinite sets which may be stated briefly as follows.

I. A set S is *finite* if it is empty or if there exists a one-to-one correspondence between S and an initial segment of the natural numbers. If S is not finite, it is *infinite*.

II. A set S is *infinite* if there exists a proper subset of S which can be put into one-to-one correspondence with S. If S is not infinite, it is *finite*.

The equivalence of these two definitions can be demonstrated (see Wilder, *Introduction to the Foundations of Mathematics*), but of course the properties of the natural numbers must be used. It is the purpose of this note to outline how some of the characteristics of finite sets can be derived directly from the second definition and then how the Peano postulates for natural numbers can be introduced easily as theorems. In what follows, therefore, we will adopt definition II when we use the words "finite" or "infinite".

Two sets, A and B, between which there is a one-to-one correspondence will be called *equivalent* and this equivalence will be denoted by $A \sim B$. Terms such as "simple order", "well-ordering", and common symbols of set theory will be presumed familiar to the reader. In particular, those terms which are not explicitly defined here will have the meaning given in the text mentioned above.

Let S be a set simply ordered by a relation "$<$". If x and x' belong to S and $x<x'$, then x' is the *successor* of x if there exists no element y of S such that $x<y<x'$. If x' is the successor of x, then x is the *predecessor* of x'. z will be called a *maximal element* of S if there is no x of S such that $z<x$, and a will be called a *minimal element* if there is no x of S such that $x<a$. S will be *discrete* if every element which is not a maximal element has a successor and if every element which is not a minimal element has a predecessor.

Theorem 1. *Every subset of a finite set is finite.*

Proof. Let A be a non-empty finite set and suppose A' is an infinite proper subset of A. Denote $A\text{-}A'$ by A''. Since A' is infinite, there exists a proper subset, $\bar{A}'$, of A' such that $\bar{A}' \sim A'$. Then $\bar{A}' \cup A'' \sim A$. But $\bar{A}' \cup A''$

is properly contained in A and therefore A is infinite. This is contrary to assumption and the theorem is proved.

Theorem 2. *Every non-empty finite set can be simply ordered. Furthermore, there exists a maximal element with respect to this simple ordering.*

Proof. Let A be a non-empty finite set and choose any element a belonging to A. If $B = A - \{a\}$, then either B is empty or contains an element, b. Let $C = B - \{b\}$ and continue in this manner so as to obtain a collection of elements,

$$S = \{a, b, c, \cdots\}$$

and a collection of subsets,

$$S' = \{A, B, C, \cdots\}$$

Pairing each element of S with the set from which it was chosen, we see that $S \sim S'$.

S' is simply ordered by "proper containment", for if X, Y, Z are members of S' then (i) if $X \neq Y$, then $X \subset Y$ or $Y \subset X$; (ii) if $X \subset Y$, then $X \neq Y$; (iii) if $X \subset Y$ and $Y \subset Z$, then $X \subset Z$. Let x, y of S be the elements associated with X, Y of S' and define the relation "$<$" such that $x < y$ if and only if $X \supset Y$. It follows that "$<$" is a simple ordering of S.

Now the process of constructing S and S' cannot continue indefinitely. Otherwise the sets S and $S'' = \{b, c, \cdots\}$ could be made equivalent by pairing every element of S with its successor in S''; this is impossible since S must be finite. Hence S must exhaust A: that is, S' contains a set Z with an element z such that $Z - \{z\}$ is empty. z is the maximal element of S.

Theorem 3. *Every finite set has a minimal and a maximal element with respect to every simple ordering of the set.*

Proof. Let A be a finite set and "$<$" any simple ordering of A. Choose any element a belonging to A. Then either a is a minimal element of A or there exists an element b belonging to $A - \{a\}$ and such that $b < a$. Again, either b is a minimal element of $A - \{a\}$ or there exists an element c belonging to $A - \{a, b\}$ such that $c < b < a$. However, as in the proof of the previous theorem the process cannot continue indefinitely for otherwise A would contain an infinite subset.

A similar argument can be made for the existence of a maximal element.

Corollary. *Every simple ordering of a finite set is a well-ordering of the set.*

Proof. Every subset of a finite set is finite and hence has a minimal element.

Theorem 4. *Every simple ordering of a finite set is discrete.*

Proof. Let a be any element (except the minimal one) of a finite set A, and let

$$L_a = \{x | x \in A, x < a\}.$$

Then L_a is a non-empty finite set and hence contains a maximal element b. Now suppose there exists a c belonging to A such that $b < c < a$. Then c is a member of L_a and $b < c$. This contradiction proves the theorem.

We now restrict our considerations to any member, S, of a class of sets which satisfy the following hypotheses.

P1. S is simply ordered by "<".

P2. S has no maximal element.

P3. If a belongs to S, $L_a = \{x | x \in S, x < a\}$ is finite.

Theorem 5. *S has a minimal element, x_0.*

Theorem 6. *S is well-ordered with respect to "<".*

Proof. Suppose S is not well-ordered. Then there exists a non-empty subset, A, of S which has no minimal element. For every y belonging to A, there is at least one element x belonging to A such that $x < y$. Therefore, L_y is not empty and contains at least one element of A. Since L_y is finite, so is $L_y \cap A$. Hence $L_y \cap A$ has a minimal element, y_0. But then y_0 is the minimal element of A, and the theorem is proved by *reductio ad absurdum*.

In the same manner as Theorem 4, we can prove

Theorem 7. *The ordering "<" of S is discrete, so that every element of S has a unique successor and, if it is not a minimal element, a unique predecessor.*

Theorem 8. (The principle of finite induction). *Let S' be any subset of S such that* (1) x_0 *belongs to* S' *and* (2) S' *contains* x', *the successor of* x, *whenever it contains* x, *then* $S' = S$.

Proof. Suppose that $S' \neq S$ and let $S'' = S - S'$. Then S'' has a minimal element a and L_a has a maximal element b. But it follows from the argument made in the proof of theorem 4 that $a = b'$ and hence a belongs to S'. This is impossible.

Theorems 5, 7, and 8 are essentially the Peano postulates.

Merrimack College

PROBLEMS AND QUESTIONS

Edited by

Robert E. Horton

Readers of this department are invited to submit for solution problems believed to be new and subject matter questions that may arise in study, in research, or in extra-academic situations. Proposals should be accompanied by solutions, when available, and by any information that will assist the editor. Ordinarily, problems in well-known textbooks should not be submitted.

Solutions should be submitted on separate, signed sheets. Figures should be drawn in India ink and twice the size desired for reproduction.

Send all communications for this department to *Robert E. Horton, Los Angeles City College, 855 North Vermont Ave., Los Angeles 29, California.*

PROPOSALS

369. *Proposed by M. S. Klamkin, AVCO, Lawrence, Massachusetts.*

An explorer travels on the surface of the earth, assumed to be a perfect sphere, in the manner to be described. First, he travels 100 miles due north. He then travels 100 miles due east. Next he travels 100 miles due south. Finally, he travels 100 miles due west, ending at the point from which he started. Determine all the possible points from which he could have started.

370. *Proposed by D. L. Silverman, Greenbelt, Maryland.*

Let xy denote x's statement to y. Determine the truth or falsity of the following set of statements :

AB : Someone is not lied to.
AC : Someone lies twice.
BA : Someone neither lies twice nor is lied to twice.
BC : Someone is lied to twice.
CA : Someone lies and is lied to.
CE : Someone does not lie.

371. *Proposed by Leon Bankoff, Los Angeles, California.*

1) Find the radius of the circle which is tangent to two internally tangent circles and to their line of centers.
2) Construct the required circle, confining all construction lines within the larger circle.

372. *Proposed by Huseyin Demir, Kandilli, Eregli, Kdz, Turkey.*

Prove the identity

$$\sin^2(\theta_1+\theta_2+\cdots+\theta_n) = \sin^2\theta_1+\cdots+\sin^2\theta_n+2\sum_{1\leqq i<j\leqq n}$$

$$\sin\theta_i\sin\theta_j\cos(\theta_1+2\theta_{i+1}+\cdots+2\theta_{j-1}+\theta_j).$$

373. *Proposed by Edgar Karst, Endicott, New York.*

Prove or disprove the statement: "If $2100n+x$ is prime, then x is prime where x is a two digit number, n is a natural number, and 01 is considered as prime.

374. *Proposed by Victor Thebault, Tennie, Sarthe, France.*

If an arbitrary straight line d, passing through any point P of the plane of a triangle ABC, meets the straight lines BC, CA and AB in points A_1, B_1 and C_1, and the points obtained in prolonging the segments A_1P, B_1P, and C_1P by three times their length are A_1', B_1', and C_1', then the mid-points of AA_1', BB_1' and CC_1', A_2, B_2, and C_2, respectively, are the vertices of a triangle, the area of which is equal to that of triangle ABC.

375. *Proposed by D.A.Steinberg, University of California Radiation Laboratory.*

Let m and n be positive integers.

1) If mn is odd then:

$$1+\sum_{k=1}^{\frac{mn-1}{2}}\sum_{j=1}^{k}\binom{n}{j}\binom{k-1}{j-1} = \frac{(m+1)^n}{2}$$

2) If mn is even then:

$$1+\sum_{k=1}^{\frac{mn-2}{2}}\sum_{j=1}^{k}\binom{n}{j}\binom{k-1}{j-1}+1/2\sum_{j=1}^{\frac{mn}{2}}\binom{n}{j}\binom{\frac{mn-2}{2}}{j-1} = \frac{(m+1)^n}{2}$$

SOLUTIONS

Errata. In problem 357, Vol. 32, No. 2, p. 105 the letter K on the right hand side of the equation and in the inequality should be a capital letter to indicate that it is different from the k in the summation.

In problem 332, Vol. 32, No. 1, p. 54 the name of *C.W.Trigg, Los Angeles City College* was omitted from the list of solvers.

The World Series

348. [September 1958] *Proposed by J. M. Howell, Los Angles City College.*

If the probability of a team winning a game in a world series is p, what is the probability of that team winning the series? What is the probability of winning in 4, 5, 6, or 7 games?

Solution by Michael J. Pascual, Siena College, New York. The probability of winning in n games is the probability of winning exactly three of the first $n-1$ games and the nth game. Denoting this probability by P_n we have by the laws of probability that

$$P_n = \binom{n-1}{3} p^3(1-p)^{n-4}\, p = \binom{n-1}{3} p^4(1-p)^{n-4}$$

hence

$$P_4 = \binom{3}{3} p^4(1-p)^0 = p^4$$

$$P_5 = \binom{4}{3} p^4(1-p) = 4p^4(1-p)$$

$$P_6 = \binom{5}{3} p^4(1-p)^2 = 10p^4(1-p)^2$$

$$P_7 = \binom{6}{3} p^4(1-p)^3 = 20p^4(1-p)^3$$

and the probability of winning the series is

$$\sum_{n=4}^{7} P_n = p^4[1+4(1-p)+10(1-p)^2+20(1-p)^3]$$

Note: the generalization of this to the case of two teams playing for the best m out of $2m-1$ is easily obtained. Letting $P_n(m)$ denote the probability of the team with probability p of winning a game, winning in n games, we easily obtain

$$P_n(m) = \binom{n-1}{m-1} p^{m+1}(1-p)^{m-n-1}, \quad n \geqq m$$

so that the probability of winning the series is

$$\sum_{n=m}^{2m-1} P_n(m)$$

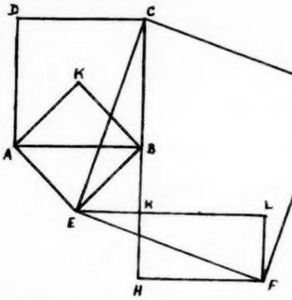
Also solved by D. A. Breault, Sylvania Electric Products, Inc., Waltham, Massachusetts; J. W. Clawson, Collegeville, Pennsylvania; Sam Kravitz, East Cleveland, Ohio; Leo Moser, University of Alberta; F. D. Parker, University of Alaska; A. Spinak, Douglas Aircraft Co., Los Angeles, California, and the proposer.

A Bisector

349. [September 1958] *Proposed by Huseyin Demir, Kandilli, Eregli, Kdz, Turkey.*

If $ABCD$, $AEBK$ and $CEFG$ are squares of the same orientations, prove that B bisects DF.

Solution by Leon Bankoff, Los Angeles, California. Removing angle CEB from the right angles AEB and CEF, we find that angles BEF, AEC, DEB are equal. But $DE = EC = EF$. Hence the triangles BDE and FBE are congruent and $FB = BD$. The collinearity of F, B, D is established by the fact that angle $EBD = 90°$.



Also solved by Norman Anning, Alhambra, California; D.A.Breault, Sylvania Electric Products, Inc., Waltham, Massachusetts; J.W.Clawson, Collegeville, Pennsylvania; Norbert Jay, New York, New York; Joseph D.E.Konhauser, Haller, Raymond and Brown, Inc., State College, Pennsylvania; Arne Pleijel, Trollhattan, Sweden; William Sanders, Mississippi Southern College; C.W.Trigg, Los Angeles City College, Dale Woods, Idaho State College, and the proposer.

The Absolute Value of x

350. [September 1958] *Proposed by George Bergman, Stuyvesant High School, New York.*

Prove that

$$|x| = x\left(\int_0^x \frac{2e^{1/t}}{(e^{1/t}+1)^2t^2}\,dt - \frac{2}{e^x+1} + 1\right)$$

for all real non zero x.

Solution by R.G.Buschman, Oregon State College.

$$\int_0^x \frac{2e^{1/t}dt}{(e^{1/t}+1)t^2} = \begin{cases} \dfrac{2}{e^{1/x}+1} - \lim\limits_{t\to 0} \dfrac{2}{e^{1/t}+1} = \dfrac{2}{e^{1/x}+1}, & x>0 \\ \dfrac{2}{e^{1/x}+1} - \lim\limits_{t\to 0-} \dfrac{2}{e^{1/t}+1} = \dfrac{2}{e^{1/x}+1} - 2, & x<0 \end{cases}$$

then

$$x\left[\int_0^x \frac{2e^{1/t}dt}{(e^{1/t}+1)^2t^2} - \frac{2}{e^{1/x}+1} + 1\right] = \begin{Bmatrix} x(+1), \ x>0 \\ x(-1), \ x<0 \end{Bmatrix} = |x|,$$

or by similar reasoning

$$x\left[\int_0^{1/x} \frac{2e^{1/t}dt}{(e^{1/t}+1)^2t^2} - \frac{2}{e^{x}+1} + 1\right] = |x|$$

Hence it appears that there is a misprint of x for $1/x$ either in the upper limit of the integral or in the second term within the parentheses.

Also solved by Stephen A. Andrea, Oberlin, Ohio; F. C. Barnett, Purdue University; E. E. Moyers, University of Mississippi; F. D. Parker, University of Alaska; Santo D. Pratico, Iona College, New York; Dale Woods, Idaho State College, and the proposer.

A Legendre Polynomial

352. [September 1958] *Proposed by L. Carlitz, Duke University.*

If $P_n(x)$ is the Legendre polynomial, show that:

I. The coefficient of x^n in the polynomial

$$(1-x^2)^n P_n\left(\frac{1+x}{1-x}\right) \quad \text{is equal to} \quad \sum_{r=0}^{n} \binom{n}{r}^3$$

II. The coefficient of x^n in the polynomial

$$(1-x)^{2n} P_n\left(\frac{1+x}{1-x}\right)^2 \quad \text{is equal to} \quad \sum_{r=0}^{n} \binom{n}{r}^4$$

Solution by Chih-yi Wang, University of Minnesota. It suffices to show

$$F(x) \equiv (1-x)^n P_n\left(\frac{1+x}{1-x}\right) = \sum_{k=0}^{n} x^k \binom{n}{k}^2$$

and the respective results follows. (For

$$(1+x)^n F(x) = \left[\sum_{r=0}^{n} x^{n-r}\binom{n}{n-r}\right]\left[\sum_{k=0}^{n} x^k \binom{n}{k}^2\right]$$

and

$$[F(x)]^2 = \left[\sum_{r=0}^{n} x^{n-r}\binom{n}{n-r}^2\right]\left[\sum_{k=0}^{n} x^k \binom{n}{k}^2\right].)$$

By substituting $z = (1+x)/(1-x)$ into the formula

$$P_n(z) = \sum_{r=0}^{\infty} \frac{(n+1)(n+2)\cdots(n+r)(-n)(1-n)\cdots(r-1-n)}{(r!)^2}\left(\frac{1}{2}-\frac{1}{2}z\right)^r$$

(see Whittaker and Watson, Modern Analysis, 4-th ed., p. 312) we obtain

$$\begin{aligned}
F(x) &= \sum_{r=0}^{n} \binom{n+r}{2r}\binom{2r}{r} x^r (1-x)^{n-r} \\
&= \sum_{r=0}^{n}\sum_{p=0}^{n-r} \binom{n+r}{2r}\binom{2r}{r}\binom{n-r}{n-r-p}(-1)^{n-r-p} x^{n-p} \\
&= \sum_{p=0}^{n} x^{n-p} \sum_{r=0}^{n-p} \binom{n+r}{2r}\binom{2r}{r}\binom{n-r}{n-r-p}(-1)^{n-r-p} \\
&= \sum_{k=0}^{n} x^k \sum_{r=0}^{k} \binom{n+r}{2r}\binom{2r}{r}\binom{n-r}{k-r}(-1)^{k-r} \\
&= \sum_{k=0}^{n} x^k \sum_{r=0}^{k} \frac{(n+r)!}{(r!)^2(k-r)!(n-k)!}(-1)^{k-r} \\
&= \sum_{k=0}^{n} x^k \binom{n}{k}^2,
\end{aligned}$$

for, by aid of the general formula

$$\binom{x}{\nu}\binom{x}{n} = \sum_{m=0}^{n} \binom{\nu+m}{\nu}\binom{x+m}{\nu+m}\binom{-m-1}{n-m}$$

(see Jordan, Calculus of Finite Differences, p. 79) we have

$$\binom{n}{n-k}\binom{n}{k} = \sum_{r=0}^{k} \binom{n-k+r}{n-k}\binom{n+r}{n-k+r}\binom{-r-1}{k-r} = \sum_{r=0}^{k} \binom{n-k+r}{n-k}\binom{n+r}{n-k+r}\binom{k}{k-r}(-1)^{k-r}$$

$$= \sum_{r=0}^{k} \frac{(n+r)!}{(r!)^2(k-r)!(n-k)!}(-1)^{k-r}.$$

Also solved by Arthur E. Danese, Union College, New York, and the proposer.

Tangent Circles

353. [September 1958] *Proposed by Karl M. Herstein, New York City, New York.*

Given a line and two points not on the line. Construct two equal circles whose centers are on the given line, which pass through the given points and are tangent to each other.

Solution by Huseyin Demir, Kandilli, Eregli, Kdz, Turkey. Let A, B, and d be the given points and the line. We distinguish two cases:

(1) The circles touch each other externally. Since the radii are equal there are no solutions except when:

(a) The circles coincide. The coincident circles contain both A and B and the center is the intersection of d and the medial line of AB.

(b) The point L of tangency is at infinity: In that case the solution consists of the perpendiculars to d from A and B.

(2) The circles touch each other internally. The solutions, if they exist, must be different from (1a) and (1b).

Take d as the x-axis and let $A(-u, a)$, $B(u, b)$ and $L(\lambda, 0)$. The circles contain the reflections of A, B with respect to d and we may there-suppose $a \geqq b > 0$.

Let the circles intersect d at $A'(\alpha, 0)$, $L(\lambda, 0)$ and L, $B'(\beta, 0)$. We have from the right triangles $A'AL$, LBB':

$$a^2 = (\lambda+u)(-u-\alpha) \qquad b^2 = (\beta-u)(u-\lambda)$$

$$\alpha = -\frac{a^2}{\lambda+u} - u \qquad \beta = -\frac{b^2}{\lambda-u} + u$$

Equating the diameters $(\lambda-\alpha)$ and $(\beta-\lambda)$ we get a cubic equation

$$2\lambda^3 + (a^2+b^2-2u^2)\lambda - u(a^2-b^2) = 0$$

Substituting $a^2 - b^2 = 2c^2$, it reduces to

$$\lambda^3 + (b^2+c^2-u^2)\lambda - uc^2 = 0$$

There are one, two (equal), or three solutions according as the discriminant Δ is positive, zero, or negative.

Now we find the relation for which

$$\Delta = 4p^3 + 27q^2 = 4(b^2+c^2-u^2)^3 + 27u^2c^4 \leqq 0$$

where $p = b^2 + c^2 - u^2$ is necessarily not positive. Hence,

$$b^2 + c^2 \leqq u^2 \quad \text{or} \quad b^2 + c^2 = u^2\cos^2 t \qquad \text{where } 0 \leqq t < \tfrac{1}{2}\pi$$

$$\Delta = 4(u^2\cos^2 t - u^2)^3 + 27u^2c^4 \leqq 0$$

$$-4u^6\sin^6 t + 27u^2c^4 \leqq 0$$

$$27c^4 \leqq 4u^4\sin^6 t$$

Since the quantities are not negative

$$\sqrt{27}\,c^2 \leqq 2u^2\sin^3 t \leqq 2u^2$$

We have finally

$$\sqrt{27}\sqrt{a^2-b^2} \begin{array}{ll} > 2u & \text{one real root} \\ = 2u & \text{double or triple root} \\ < 2u & \text{three real roots} \end{array}$$

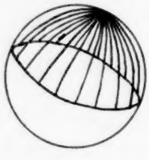
Also solved by Sam Kravitz, East Cleveland, Ohio.

The Archer

354. [September 1958] *Proposed by Lowell Van Tassel, San Diego Junior College, California.*

A spherical shell is tossed into the air and is shot at by a mathematical archer at infinity. One hemisphere of the shell is painted black. If the archer hits the shell which has been randomly spun, what is the probability that his vector-arrow has either entered or left through a blackened area? Consider the arrow point-sized, the sphere of radius unity and the probability of a hit certain.

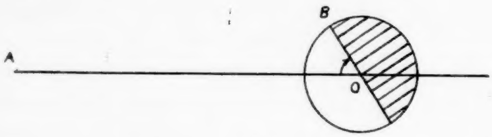
Solution by Steve Andrea, Oberlin, Ohio. If the archer is at infinity, what he sees is the projection of the shell onto a plane perpendicular to the line from the archer at A to the center of the shell O. Letting the shell be translucent, the archer sees this:



The minor axis of the ellipse may vary from zero to one.

Now if the arrow hits in the upper crescent, it will enter and leave through black; if it hits within the ellipse, it will penetrate the black either upon going in or upon going out, but not in both.

The minor axis of the ellipse equals $\sin V$, where V is the angle AOB, the smaller angle between AO and the plane which separates the shell into two hemispheres.



Now, the probability P_1 that the arrow will enter and leave through black is equal to

$$2/\pi\int_0^{\pi/2} p(V)\,dV, \text{ where } p(V) = \pi/2-\int_{-1}^{+1} \sin V\sqrt{1-x^2}\,dx.$$

Thus,

$$P_1 = \frac{\pi-2}{2\pi}$$

Next, the probability P_2 of entering or leaving in black, but not in both, equals

$$2/\pi\int_0^{\pi/2} q(V)\,dV, \text{ where } q(V) = \sin V. \text{ Here } P_2 = 2/\pi.$$

The answer to the problem is the sum of these,

$$P = P_1+P_2 = \frac{\pi+2}{2\pi}$$

Comment on Problem 336

336. [March 1958 and November 1958] *Proposed by C.W. Trigg, Los Angeles City College.*

Comment by William E.F. Appuhn, St. John's University, New York.

In the statement of the problem, the plane of the bar and the string is given merely as perpendicular to the wall, not as a vertical plane perpendicular to the wall. It is therefore necessary first to prove, rather than assume, that the plane of the string and the bar is a vertical plane.

This can be done quite readily by assuming that the plane of the string and the bar makes an angle α, $0 < \alpha \leqq \pi/2$ with the vertical and taking moments, about an axis perpendicular to the wall at the point of contact of the par with the wall, of all the forces acting on the bar. Then the force of gravity on the bar would be the only force having a moment about this axis, since the lines of action of all the other forces pass through the axis. Thus, we conclude that the plane of the string and the bar must be vertical for equilibrium to exist. We are not, until now justified in saying: "The vertical force of Friction", and that, "the line of action of the force of gravity on the bar intersects the string", etc.

QUICKIES

From time to time this department will publish problems which may be solved by laborious methods, but which with the proper insight may be disposed of with dispatch. Readers are urged to submit their favorite problems of this type, together with the elegant solution and the source, if known.

Q241. Show that the medians of a scalene triangle do not bisect any of the angles of the triangle. [*Submitted by Brother T. Brendan, F.S.C.*]

Q242. Find an $f(n)$ such that $f(\text{even}) = \frac{1}{2}$ and $f(\text{odd}) = 1$ [*Submitted by Huseyin Demir*]

Q243. Derive the formula

$$\tan \tfrac{1}{2} A = \frac{\sin A}{1 + \cos A}$$

without using radicals at any stage of the work. [*Submitted by Dick Wick Hall*]

Q244. Prove that

$$\sum_{n=1}^{\infty} 6^{\frac{2-3n-n^2}{2}}$$

is irrational. [*Submitted by David L. Silverman*]

Q245. Ten people decided to start a club. If there had been 5 more in the group, the initial expense to each would have been \$100 less. What was the initial cost per person? [*Submitted by C. W. Trigg*]

Q 246. Determine the class of angles which can be trisected with straight edge and compasses. [*Submitted by M. S. Klamkin*]

Answers

A 241. Assume that in some scalene triangle one of the medians does bisect its angle. "Folding" the triangle over on itself along that median leads to a ready contradiction.

A 242. Construct $f(n)$ so that $f(n) = \frac{1}{2}(n+1-2[n/2])$ where $[n/2]$ is the largest integer in $n/2$.

A 243. Write

$$\tan \tfrac{1}{2}A = \frac{2\sin\frac{1}{2}A\cos\frac{1}{2}A}{2\cos^2\frac{1}{2}A} = \frac{\sin A}{1+\cos A}$$

A 244. The exponents form the sequence $-1, -4, -8, -13, -19, -26, \cdots$ with difference of terms constantly increasing by 1. Written in base 6, we get the non repeating decimal $.1001000100001000001\cdots$

A 245. With 50% more to share the expense, the cost per person would have been 2/3 as great. Thus, \$100 is one-third and the initial individual cost was \$300.

A 246. Since $\cos\theta = 4\cos^3\theta/3 - 3\cos\theta/3$, it follows that angles of the form $\theta = \arccos(4x^3-3x)$ can be trisected where x is constructible and $\cos\theta/3 = x$.

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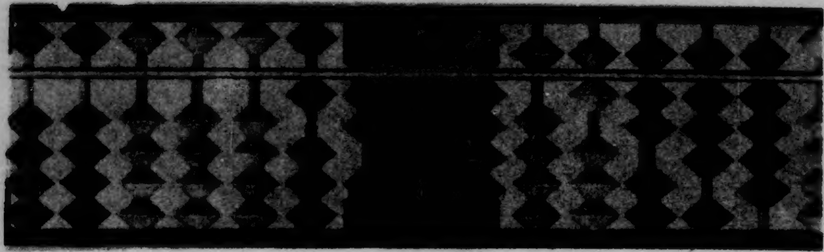

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